Linear Programming

Make sure you enroll in the department's elective course

LP

- Motivating examples
- Introduction to algorithms
- Simplex algorithm
  - On a particular example

Example 1: profit maximization

- A company has two types of products: P, Q.
- Profit: P --- \$1 each; Q --- \$6 each.
- Constraints:
  - Daily productivity (including both P and Q) is 400
  - Daily demand for P is 200
  - Daily demand for Q is 300
- *Question: How many P and Q should we produce to maximize the profit?*
  - x_1 units of P, x_2 units of Q

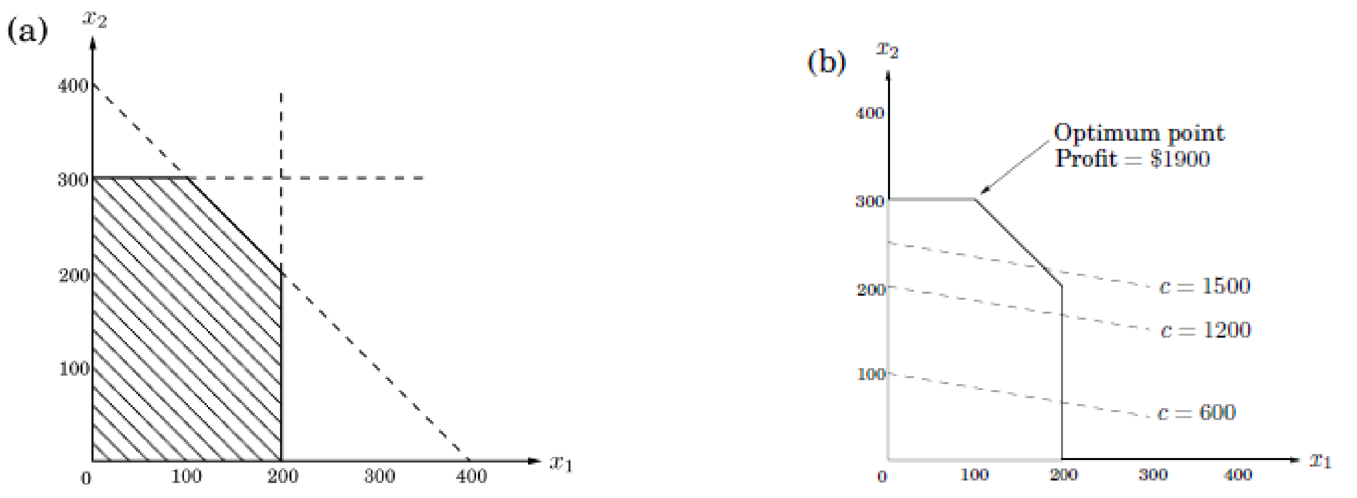
How to solve?

- x_1 units of P
  x_2 units of Q
- Constraints:
  - Daily productivity (including both P and Q) is 400
  - Daily demand for P is 200
  - Daily demand for Q is 300
- Question: how much P and Q to produce to maximize the profit?

- Variables:
  - x_1 and x_2.
- Constraints:
  - $x_1 + x_2 \leq 400$
  - $x_1 \leq 200$
  - $x_2 \leq 300$
  - $x_1, x_2 \geq 0$
- Objective:

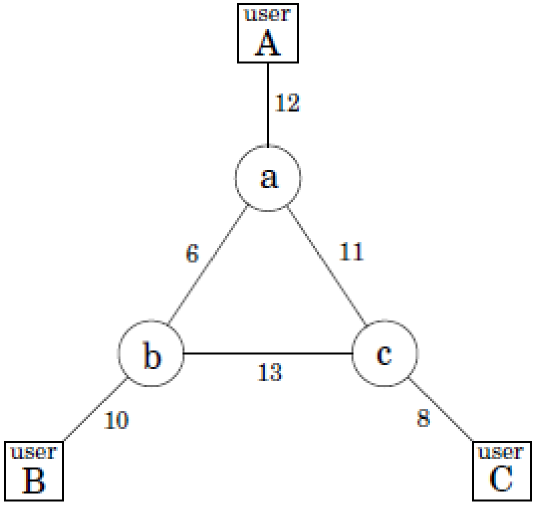
  $$\max x_1 + 6x_2$$

Illustrative figures



Example 2

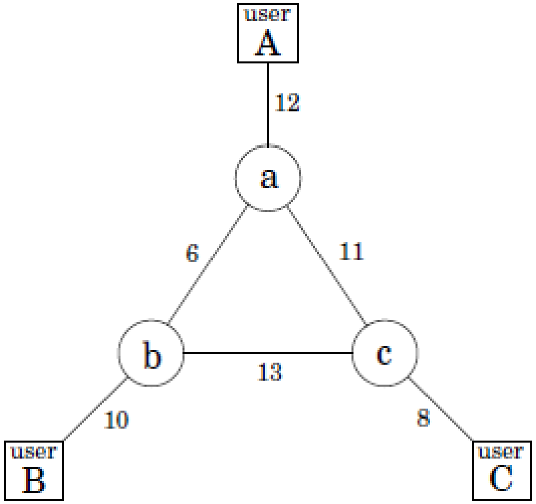
- We are managing a network with bandwidth as shown by numbers on edges.
  - Bandwidth: max units of flows
- 3 connections: AB, BC, CA
  - We get \$3, \$2, \$4 for providing them respectively.
  - Two routes for each connection: short and long.
- *Question: How to route the connections to maximize our revenue?*



Example 2

x_{AB}: amount of flow of the short route
x'_{AB}: amount of flow of the long route

- Variables:
  - $x_{AB}, x'_{AB}, x_{BC}, x'_{BC}, x_{AC}, x'_{AC}$.
- Constraints:
  - $x_{AB} + x'_{AB} + x_{AC} + x'_{AC} \leq 12$ (edge (A, a))
  - $x_{AB} + x'_{AB} + x_{BC} + x'_{BC} \leq 10$ (edge (B, b))
  - $x_{BC} + x'_{BC} + x_{AC} + x'_{AC} \leq 8$ (edge (C, c))
  - $x_{AB} + x'_{BC} + x'_{AC} \leq 6$ (edge (a, b))
  - $x'_{AC} + x'_{AB} + x_{BC} \leq 13$ (edge (b, c))
  - $x_{AB} + x'_{BC} + x'_{AC} \leq 11$ (edge (a, c))
  - $x_{AB}, x'_{AB}, x_{BC}, x'_{BC}, x_{AC}, x'_{AC} \geq 0$



- Objective:
  max $3(x_{AB} + x'_{AB}) + 2(x_{BC} + x'_{BC}) + 4(x_{AC} + x'_{AC})$

LP in general

- Max/min a linear function of variables
  - Called the *objective function*
- All constraints are linear (in)equalities
- Equational form:

Superscript ^{T}: transpose of vectors.

$$\begin{array}{lll} \text{max} & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \text{max} & c_1 x_1 + \cdots + c_n x_n \\ \text{s.t.} & a_{i1} x_1 + \cdots + a_{in} x_n = b_i, \\ & \qquad \forall i = 1, \dots, m \\ & x_i \geq 0, \ \forall i = 1, \dots, n \end{array}$$

Inequality: entry-wise

  - $\boldsymbol{x}$: variables.
  - $(A, \boldsymbol{b})$: coefficients in constraints

Transformations between forms

- Min vs. max:
  - $\min \boldsymbol{c}^T\boldsymbol{x} \Leftrightarrow \max -\boldsymbol{c}^T\boldsymbol{x}$

- Inequality directions:
  - $\boldsymbol{a}_{\boldsymbol{i}}^T\boldsymbol{x} \geq b_i \Leftrightarrow -\boldsymbol{a}_{\boldsymbol{i}}^T\boldsymbol{x} \leq -b_i$

- Equalities to inequalities: ($\boldsymbol{a}_{\boldsymbol{i}}$: row i in matrix A)
  - $\boldsymbol{a}_{\boldsymbol{i}}^T\boldsymbol{x} = b_i \Leftrightarrow \boldsymbol{a}_{\boldsymbol{i}}^T\boldsymbol{x} \geq b_i$, and $\boldsymbol{a}_{\boldsymbol{i}}^T\boldsymbol{x} \leq b_i$.

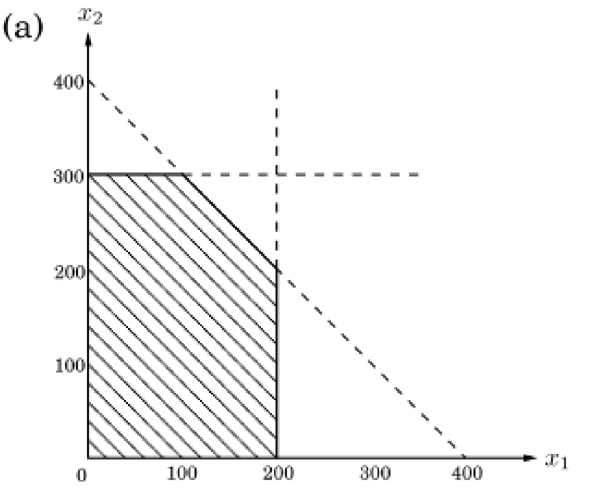
Transformations between forms

- Inequalities to equalities:
  - $\boldsymbol{a_i^T x} \geq b_i \Leftrightarrow \boldsymbol{a_i^T x} = b_i + s_i, s_i \geq 0$
    - The newly introduced variable s_i is called *slack variable*

- "Unrestricted" to "nonnegative constraint":
  - x_i unrestricted $\Leftrightarrow x_i = s - t, s \geq 0, t \geq 0$

feasibility

- The constraints of the form $ax_1 + bx_2 = c$ is a line on the plane of (x_1, x_2).
- $ax_1 + bx_2 \leq c$? half space.
  - $x_1 \leq 200$
  - $x_2 \leq 300$
  - $x_1 + x_2 \leq 400$
  - $x_1, x_2 \geq 0$

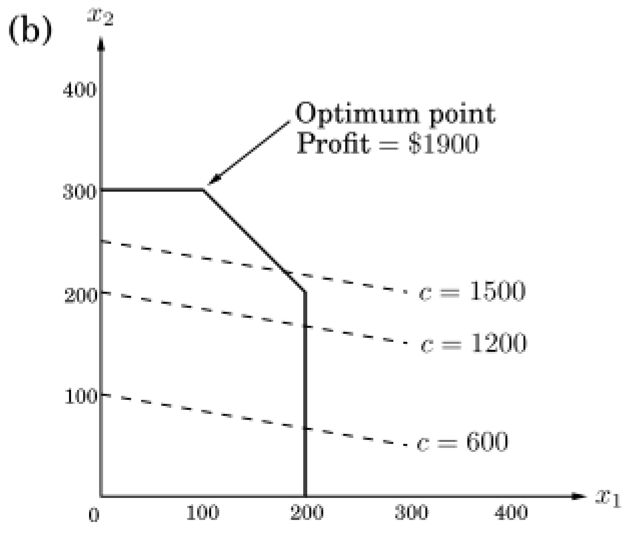


- All constraints are satisfied: the intersection of these half spaces. --- feasible region.
  - Feasible region nonempty: LP is feasible
  - Feasible region empty: LP is infeasible

Adding the objective function into the picture

- The objective function is also linear
  - also a line for a fixed value.
- Thus the optimization is:

  try to move the line towards the desirable direction s.t. the line still intersects with the feasible region.



Possibilities of solution

- Infeasible: no solution satisfying

  $$A\boldsymbol{x} = \boldsymbol{b} \text{ and } \boldsymbol{x} \geq 0.$$

  - Example? Picture?
- Feasible but unbounded: $\boldsymbol{c}^T\boldsymbol{x}$ can be arbitrarily large.
  - Example? Picture?
- Feasible and bounded: there is an optimal solution.
  - Example? Picture?

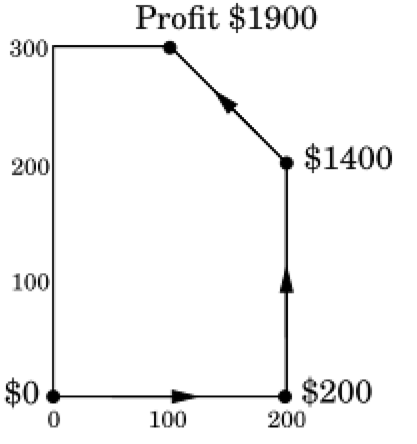
Three Algorithms for LP

- Simplex algorithm (Dantzig, 1947)
  - Exponential in worst case
  - Widely used due to the practical efficiency
- Ellipsoid algorithm (Khachiyan, 1979)
  - First polynomial-time algorithm: $O(n^4 L)$
    - L: number of input bits
  - Little practical impact.
- Interior point algorithm (Karmarkar, 1984)
  - More efficient in theory: $O(n^{3.5} L)$
  - More efficient in practice (compared to Ellipsoid).

Weakly polynomial time

Simplex method: geometric view

- Start from any vertex of the feasible region.
- Repeatedly look for a better neighbor and move to it.
  - Better: for the objective function
- Finally we reach a point with no better neighbor
  - In other words, it's locally optimal.



- For LP: locally optimal ⇔ globally optimal.
  - Reason: the feasible region is a convex set.

Simplex algorithm: Framework

- A sequence of (simplex) tableaus

1. Pick an initial tableau
2. Update the tableau
3. Terminate

- What's a tableau?

1. How?
2. What's the rule?
3. When to terminate? Why optimal?

Complexity?

An introductory example

- Consider the following LP

$$\begin{aligned} \max \quad & x_1 + x_2 \\ s.t. \quad & -x_1 + x_2 + x_3 = 1 \\ & x_1 + x_4 = 3 \\ & x_2 + x_5 = 2 \\ & x_1, \dots, x_5 \geq 0 \end{aligned}$$

- The equalities are $Ax = b$,

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

- Let $z = obj = x_1 + x_2$.

- Rewrite equalities as follows. (A tableau.)

$$\begin{aligned} x_3 &= 1 + x_1 - x_2 \\ x_4 &= 3 - x_1 \\ x_5 &= 2 - x_2 \\ z &= x_1 + x_2 \end{aligned}$$

An introductory example

- The equalities are $Ax = b$,
  $A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$
- Let $z = obj = x_1 + x_2$.
- $B = \{3,4,5\}$ is a basis:
  $A_B = I_3$ is non-singular.
  - A_B: columns $\{j : j \in B\}$ of A.
- The basis is feasible:

$$A_B^{-1} b = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

- Rewrite equalities as follows.

$$x_3 = 1 + x_1 - x_2$$
$$x_4 = 3 - x_1$$
$$x_5 = 2 - x_2$$
$$z = x_1 + x_2$$

- Set $x_1 = x_2 = 0$, and get $x_3 = 1, x_4 = 3, x_5 = 2$.
- And $z = 0$.
- $\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & z \\ 0 & 0 & 1 & 3 & 2 & 0 \end{pmatrix}$

An introductory example

- Now we want to improve $z = obj = x_1 + x_2$.
- Clearly one needs to increase x_1 or x_2.
- Let's say x_2.
  - we keep $x_1 = 0$.
- How much can we increase x_2?
  - We need to maintain the first three equalities.
- Rewrite equalities as follows.

$$x_3 = 1 + x_1 - x_2$$
$$x_4 = 3 - x_1$$
$$x_5 = 2 - x_2$$
$$z = x_1 + x_2$$

- Set $x_1 = x_2 = 0$, and get $x_3 = 1, x_4 = 3, x_5 = 2.$
- And $z = 0$.
- $\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & z \\ 0 & 0 & 1 & 3 & 2 & 0 \end{pmatrix}$

An introductory example

- Setting $x_1 = 0$, the first three equalities become
$$x_3 = 1 - x_2$$
$$x_4 = 3$$
$$x_5 = 2 - x_2$$
- To maintain all $x_i \geq 0$, we need $x_2 \leq 1$ and $x_2 \leq 2$.
  - obtained from the first and third equalities above.
- So x_2 can increase to 1.
- And x_3 becomes 0.
- Rewrite equalities as follows.
$$x_3 = 1 + x_1 - x_2$$
$$x_4 = 3 - x_1$$
$$x_5 = 2 - x_2$$
$$z = x_1 + x_2$$
- Set $x_1 = 0,\ x_2 = 1$, and update other variables $x_3 = 0, x_4 = 3, x_5 = 1.$
- And $z = 1$.
- $\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & z \\ 0 & 1 & 0 & 3 & 1 & 1 \end{pmatrix}$

An introductory example

- Now basis becomes {2,4,5}
  - the basis is feasible.
- Compare to previous basis {3,4,5}, one index (3) leaves and another (2) enters.
- This process is called a pivot step.
- Rewrite the tableau by putting variables in basis to the left hand side.

- Rewrite equalities as follows.

$$
\begin{aligned}
x_3 &= 1 + x_1 - x_2 \\
x_4 &= 3 - x_1 \\
x_5 &= 2 - x_2 \\
z &= x_1 + x_2
\end{aligned}
$$

An introductory example

- Now basis becomes {2,4,5}
  - the basis is feasible.
- Compare to previous basis {3,4,5}, one index (3) leaves and another (2) enters.
- This process is called a pivot step.
- Rewrite the tableau by putting variables in basis to the left hand side.
- Rewrite equalities as follows.

$$x_2 = 1 + x_1 - x_3$$
$$x_4 = 3 - x_1$$
$$x_5 = 1 - x_1 + x_3$$
$$z = 1 + 2x_1 - x_3$$

An introductory example

- Repeat the process.
- To increase z, we can increase x_1.
  - Increasing x_3 decreases z since the coefficient is negative.
- We keep $x_3 = 0$, and see how much we can increase x_1.
- We can increase x_1 to 1, at which point x_5 becomes 0.
- Rewrite equalities as follows.

$$
\begin{aligned}
x_2 &= 1 + x_1 - x_3 \\
x_4 &= 3 - x_1 \\
x_5 &= 1 - x_1 + x_3 \\
z &= 1 + 2x_1 - x_3
\end{aligned}
$$

- Set $x_3 = 0,\ x_1 = 1$, and update other variables $x_2 = 2, x_4 = 2, x_5 = 0$.
- And $z = 3$.
- $\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & z \\ 1 & 2 & 0 & 2 & 0 & 3 \end{pmatrix}$

An introductory example

- The new basis is $\{1,2,4\}$.
- Rewrite the tableau.

- Rewrite equalities as follows.

$$x_2 = 1 + x_1 - x_3$$
$$x_4 = 3 - x_1$$
$$x_5 = 1 - x_1 + x_3$$
$$z = 1 + 2x_1 - x_3$$

- Set $x_3 = 0,\ x_1 = 1$, and update other variables $x_2 = 2, x_4 = 2, x_5 = 0.$
- And $z = 3$.
- $\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & z \\ 1 & 2 & 0 & 2 & 0 & 3 \end{pmatrix}$

An introductory example

- The new basis is $\{1,2,4\}$.
- Rewrite the tableau.
- See which variable should increase to make z larger.
  - x_3 in this case.
- See how much we can increase x_3.
  - $x_3 = 2$.
- Update x_i's and z.
- Rewrite equalities as follows.

$$\begin{aligned} x_1 &= 1 + x_3 - x_5 \\ x_2 &= 2 - x_5 \\ x_4 &= 2 - x_3 + x_5 \\ z &= 3 + x_3 - 2x_5 \end{aligned}$$

- Set $x_5 = 0, x_3 = 2$, and update other variables $x_1 = 3, x_2 = 2, x_4 = 0$.
- And $z = 5$.
- $\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & z \\ 3 & 2 & 2 & 0 & 0 & 5 \end{pmatrix}$

An introductory example

- The new basis is $\{1,2,3\}$.
- Rewrite the tableau.
- See which variable should increase to make z larger.
- None!
  - Both coefficients for x_4 and x_5 are negative now.
- Claim: We've found the optimal solution and optimal value! ☺

- Rewrite equalities as follows.

$$\begin{aligned} x_1 &= 3 - x_4 \\ x_2 &= 2 - x_5 \\ x_3 &= 2 - x_4 + x_5 \\ z &= 5 - x_4 - x_5 \end{aligned}$$

- Set $x_5 = 0, x_3 = 2$, and update other variables $x_1 = 3, x_2 = 2, x_4 = 0.$
- And $z = 5$.
- $\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & z \\ 3 & 2 & 2 & 0 & 0 & 5 \end{pmatrix}$