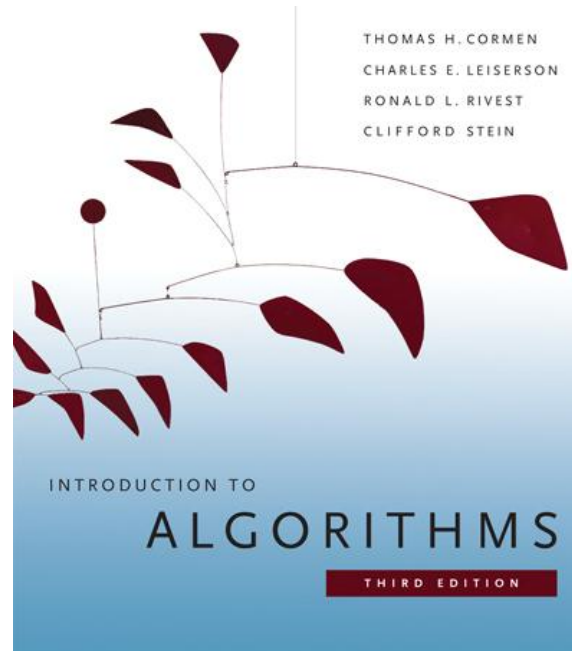


Introduction to Algorithms



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Menu

- Priority Queues
- Heaps
- Heapsort

Priority Queue

A data structure implementing a set S of elements, each associated with a key, supporting the following operations:

$\text{insert}(S, x)$: insert element x into set S

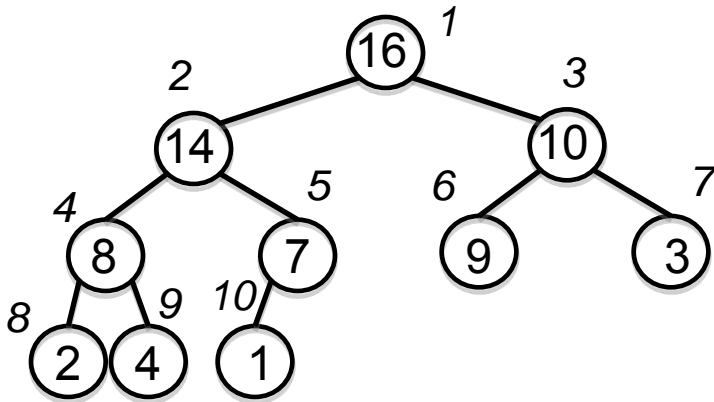
$\text{max}(S)$: return element of S with largest key

$\text{extract_max}(S)$: return element of S with largest key and remove it from S

$\text{increase_key}(S, x, k)$: increase the value of element x 's key to new value k
(assumed to be as large as current value)

Heap

- Implementation of a priority queue
- An **array**, visualized as a nearly complete **binary tree**
- **Max Heap Property**: The key of a node is \geq than the keys of its children
(**Min Heap** defined analogously)



1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

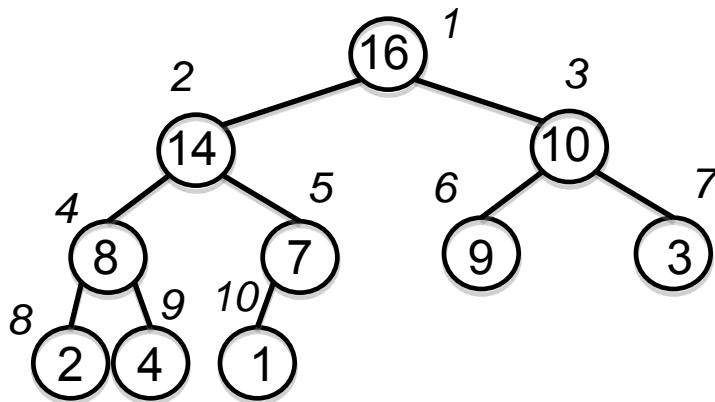
Heap as a Tree

root of tree: first element in the array, corresponding to $i = 1$

$\text{parent}(i) = i/2$: returns index of node's parent

$\text{left}(i) = 2i$: returns index of node's left child

$\text{right}(i) = 2i+1$: returns index of node's right child



1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

No pointers required! Height of a binary heap is $O(\lg n)$

Heap Operations

`build_max_heap` : produce a max-heap from an unordered array

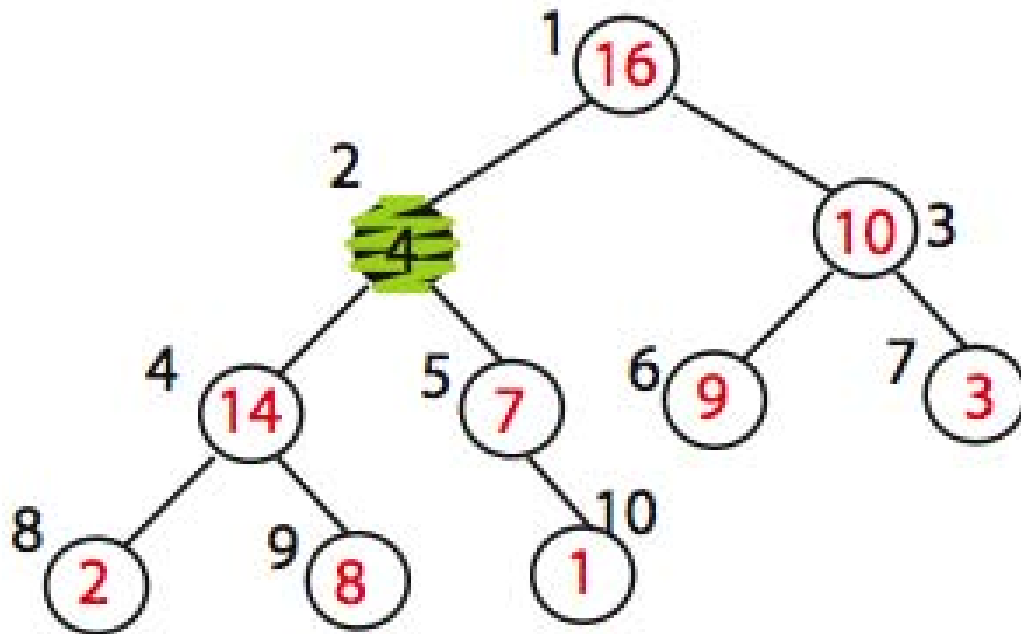
`max_heapify` : correct a **single** violation of the heap property in a subtree at its root

`insert, extract_max, heapsort`

Max_heapify

- Assume that the trees rooted at $\text{left}(i)$ and $\text{right}(i)$ are max-heaps
- If element $A[i]$ violates the max-heap property, correct violation by “trickling” element $A[i]$ down the tree, making the subtree rooted at index i a max-heap

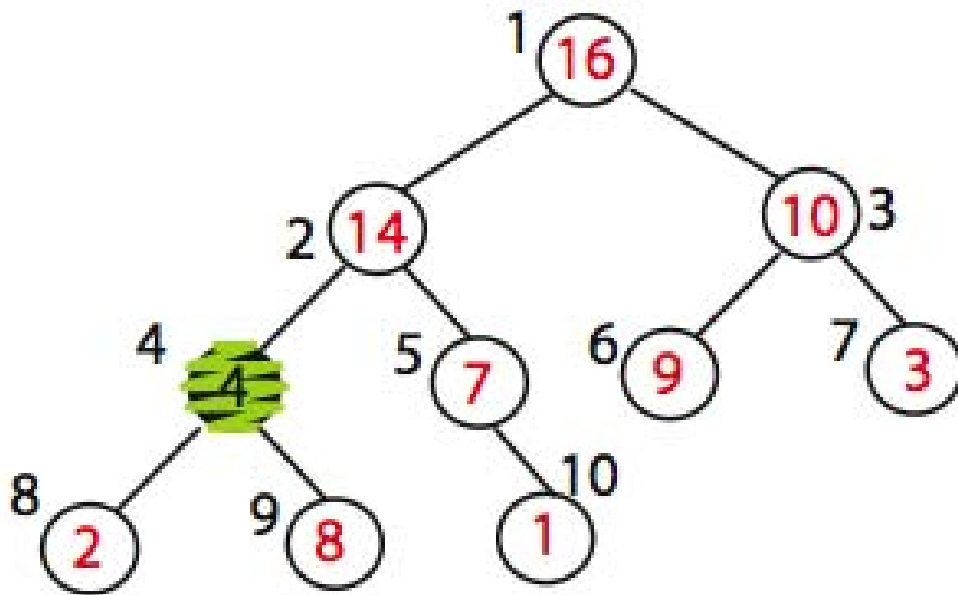
Max_heapify (Example)



MAX_HEAPIFY (A,2)
heap_size[A] = 10

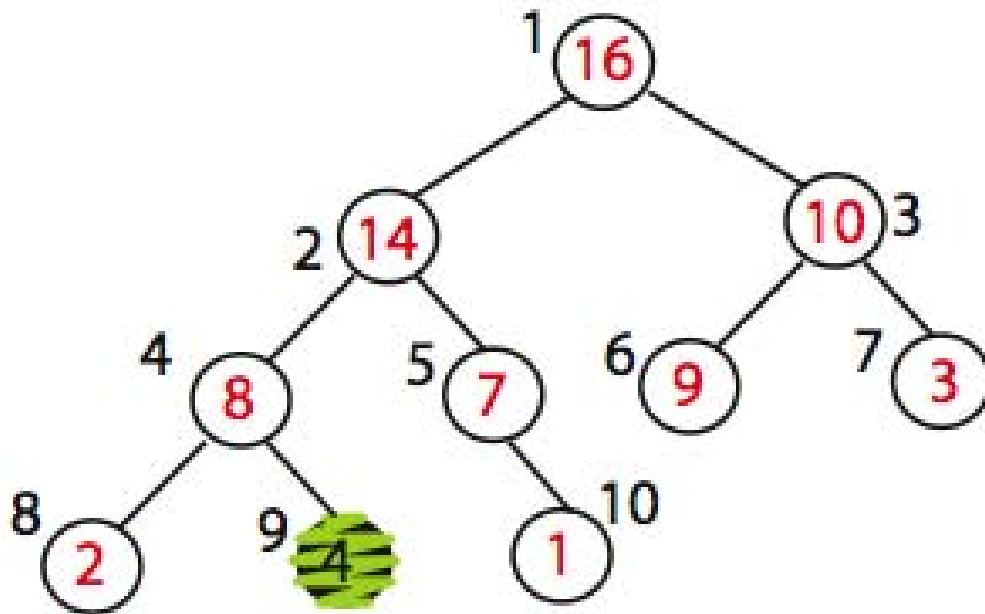
Node 10 is the left child of node 5 but is drawn to the right for convenience

Max_heapify (Example)



Exchange A[2] with A[4]
Call MAX_HEAPIFY(A,4)
because max_heap property
is violated

Max_heapify (Example)



Exchange A[4] with A[9]
No more calls

Time=? $O(\log n)$

Max_Heapify Pseudocode

$l = \text{left}(i)$

$r = \text{right}(i)$

if ($l \leq \text{heap-size}(A)$ and $A[l] > A[i]$)

 then $\text{largest} = l$ else $\text{largest} = i$

if ($r \leq \text{heap-size}(A)$ and $A[r] > A[\text{largest}]$)

 then $\text{largest} = r$

if $\text{largest} \neq i$

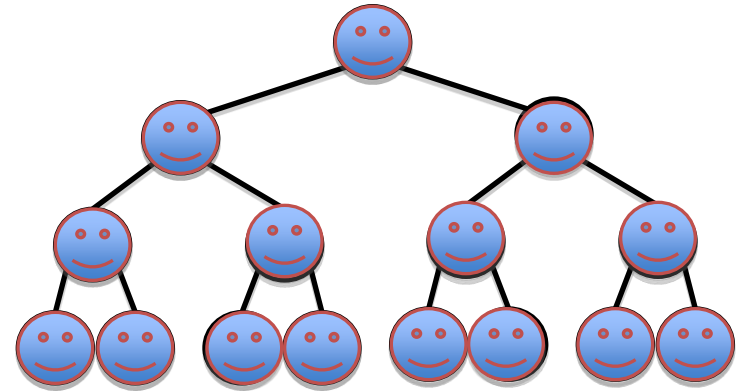
 then exchange $A[i]$ and $A[\text{largest}]$

 Max_Heapify($A, \text{largest}$)

Build_Max_Heap(A)

Converts $A[1 \dots n]$ to a max heap

```
Build_Max_Heap(A):  
  for  $i = n/2$  downto 1  
    do Max_Heapify(A, i)
```



Why start at $n/2$?

Because elements $A[n/2 + 1 \dots n]$ are all leaves of the tree

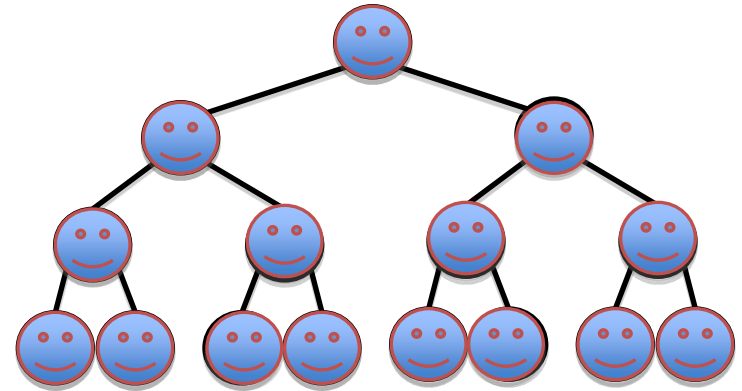
$2i > n$, for $i > n/2 + 1$

Time=? $O(n \log n)$ via simple analysis

Build_Max_Heap(A) Analysis

Converts $A[1 \dots n]$ to a max heap

```
Build_Max_Heap(A):  
  for  $i = n/2$  downto 1  
    do Max_Heapify(A, i)
```

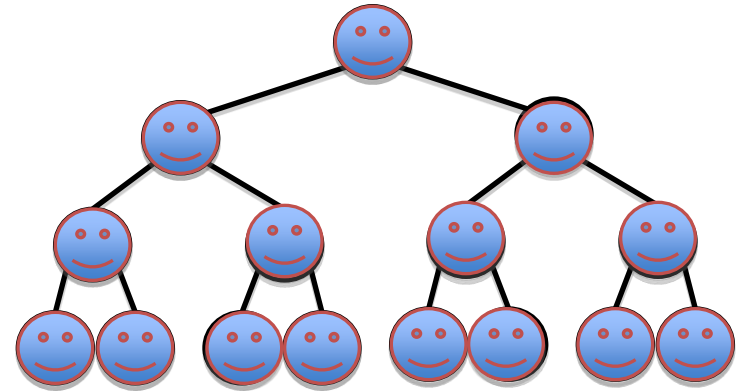


Observe however that Max_Heapify takes $O(1)$ for time for nodes that are one level above the leaves, and in general, $O(l)$ for the nodes that are l levels above the leaves. We have $n/4$ nodes with level 1, $n/8$ with level 2, and so on till we have one root node that is $\lg n$ levels above the leaves.

Build_Max_Heap(A) Analysis

Converts $A[1 \dots n]$ to a max heap

```
Build_Max_Heap(A):  
  for i=n/2 downto 1  
    do Max_Heapify(A, i)
```



Total amount of work in the for loop can be summed as:

$$n/4 (1 c) + n/8 (2 c) + n/16 (3 c) + \dots + 1 (\lg n c)$$

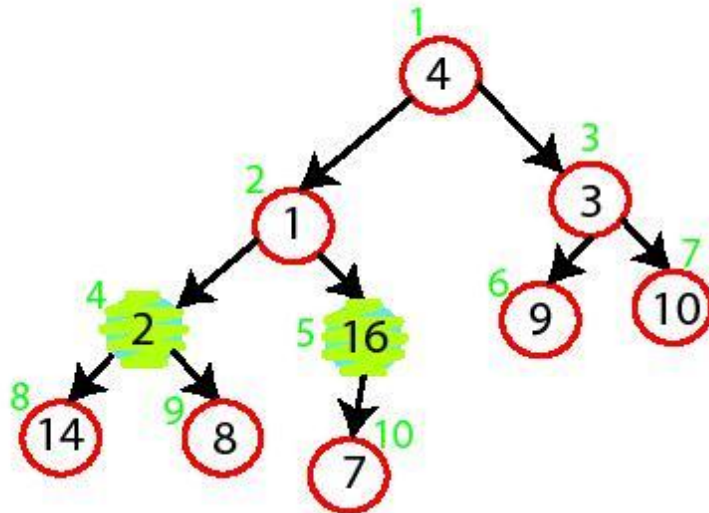
Setting $n/4 = 2^k$ and simplifying we get:

$$c 2^k (1/2^0 + 2/2^1 + 3/2^2 + \dots (k+1)/2^k)$$

The term in brackets is bounded by a constant!

This means that Build_Max_Heap is $O(n)$

Build-Max-Heap Demo



A

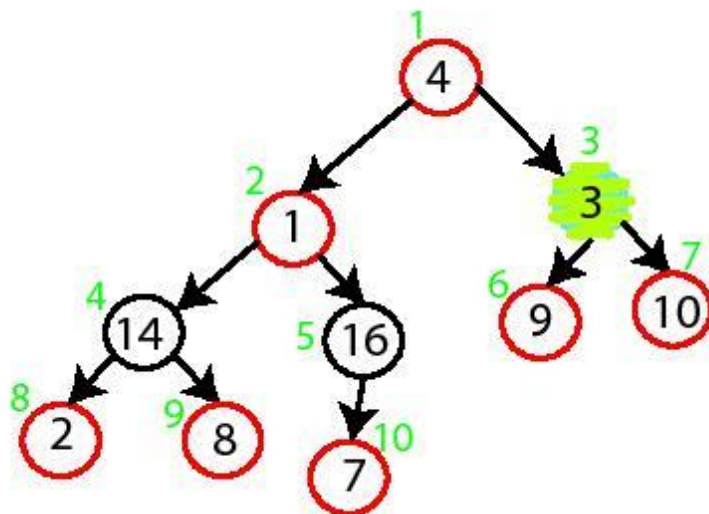
4	1	3	2	16	9	10	14	8	7
---	---	---	---	----	---	----	----	---	---

MAX-HEAPIFY (A,5)

no change

MAX-HEAPIFY (A,4)

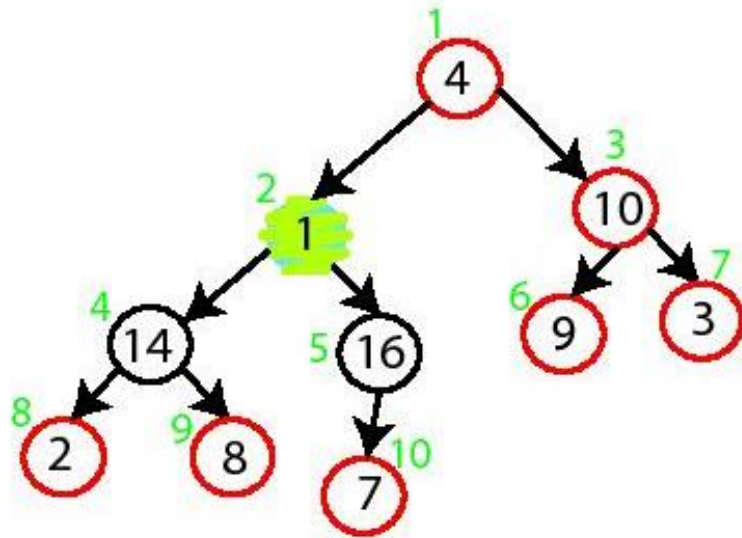
Swap A[4] and A[8]



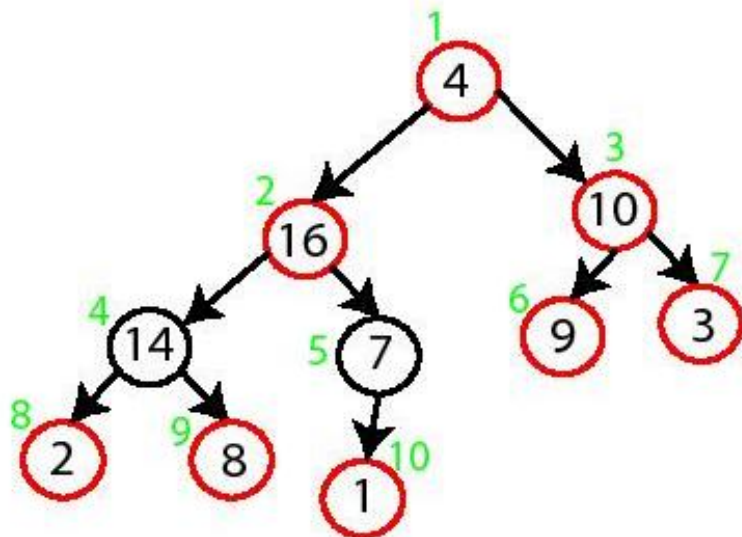
MAX-HEAPIFY (A,3)

Swap A[3] and A[7]

Build-Max-Heap Demo

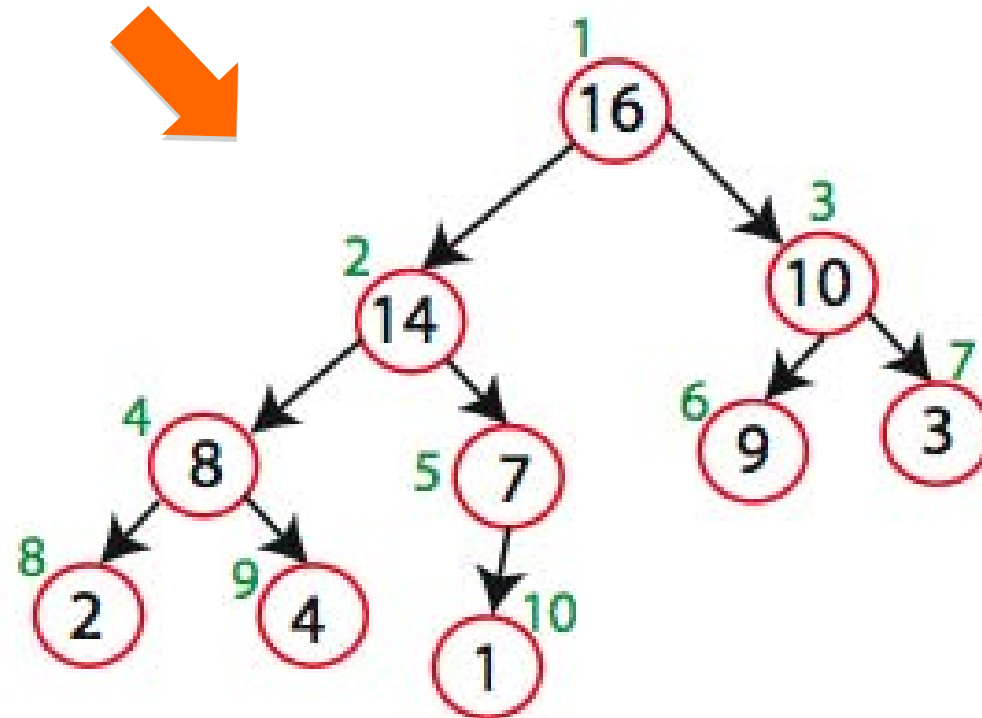


MAX-HEAPIFY (A,2)
Swap A[2] and A[5]
Swap A[5] and A[10]



MAX-HEAPIFY (A,1)
Swap A[1] with A[2]
Swap A[2] with A[4]
Swap A[4] with A[9]

Build-Max-Heap



Heap-Sort

Sorting Strategy:

1. Build Max Heap from unordered array;

Heap-Sort

Sorting Strategy:

1. Build Max Heap from unordered array;
2. Find maximum element $A[1]$;
3. Swap elements $A[n]$ and $A[1]$:
now max element is at the end of the array!

Heap-Sort

Sorting Strategy:

1. Build Max Heap from unordered array;
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now max element is at the end of the array!
4. Discard node n from heap
(by decrementing heap-size variable)

Heap-Sort

Sorting Strategy:

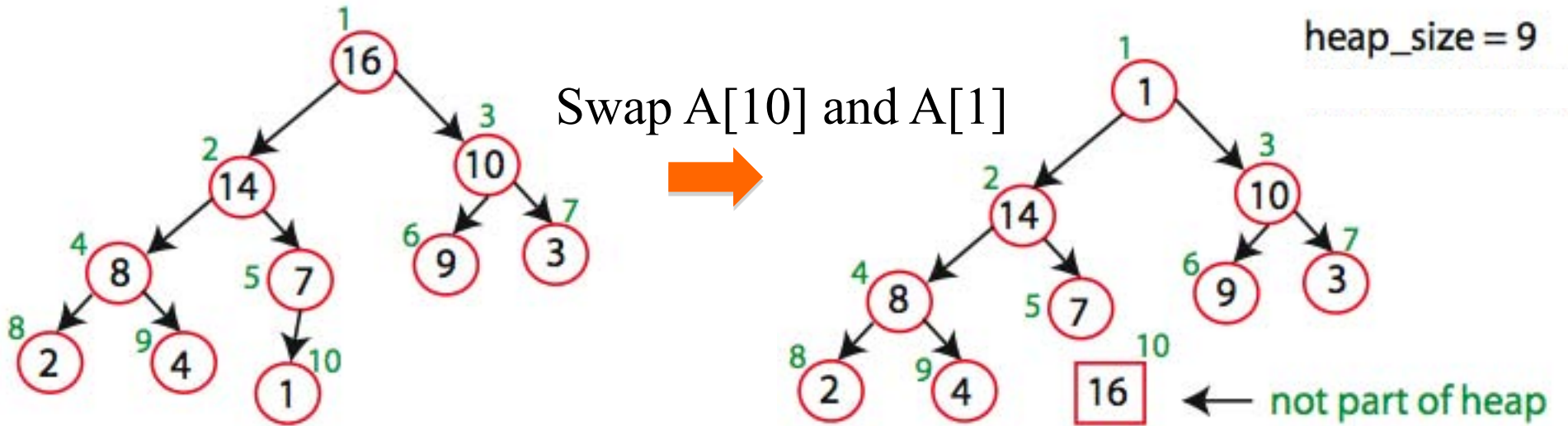
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(by decrementing heap-size variable)
5. New root may violate max heap property, but its children are max heaps. Run `max_heapify` to fix this.

Heap-Sort

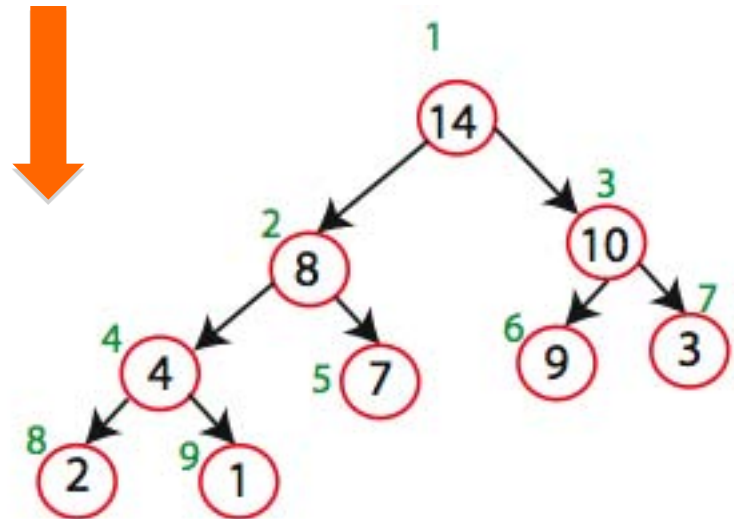
Sorting Strategy:

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4. Discard node n from heap
(by decrementing heap-size variable)
5. New root may violate max heap property, but its children are max heaps. Run `max_heapify` to fix this.
6. Go to Step 2 unless heap is empty.

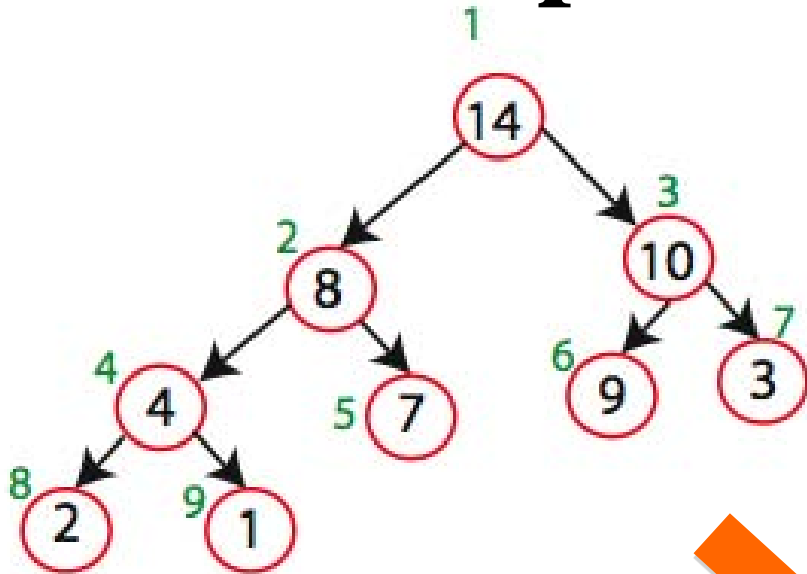
Heap-Sort Demo



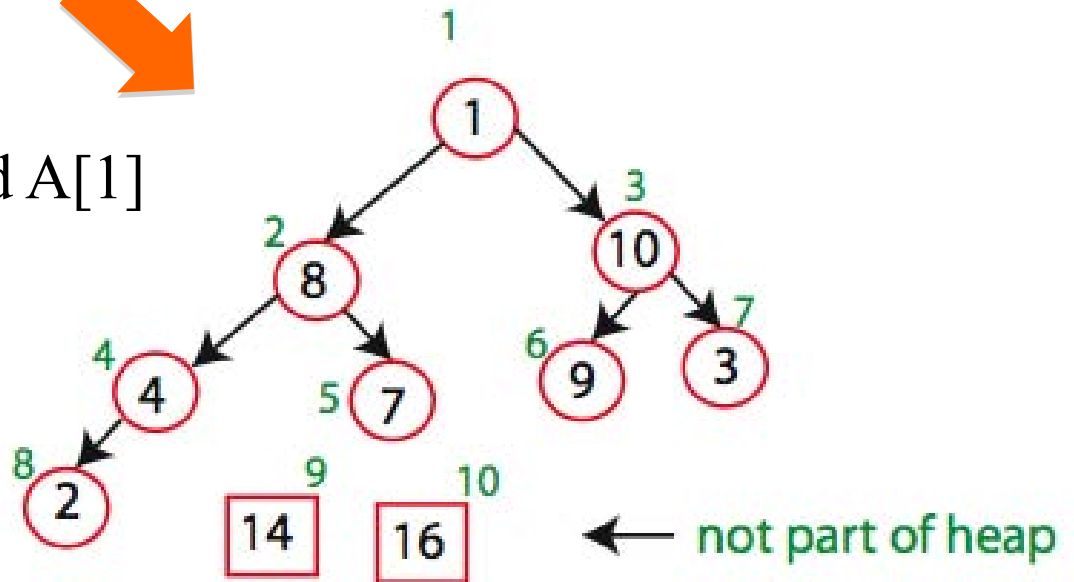
Max_heapify(A,1)



Heap-Sort Demo

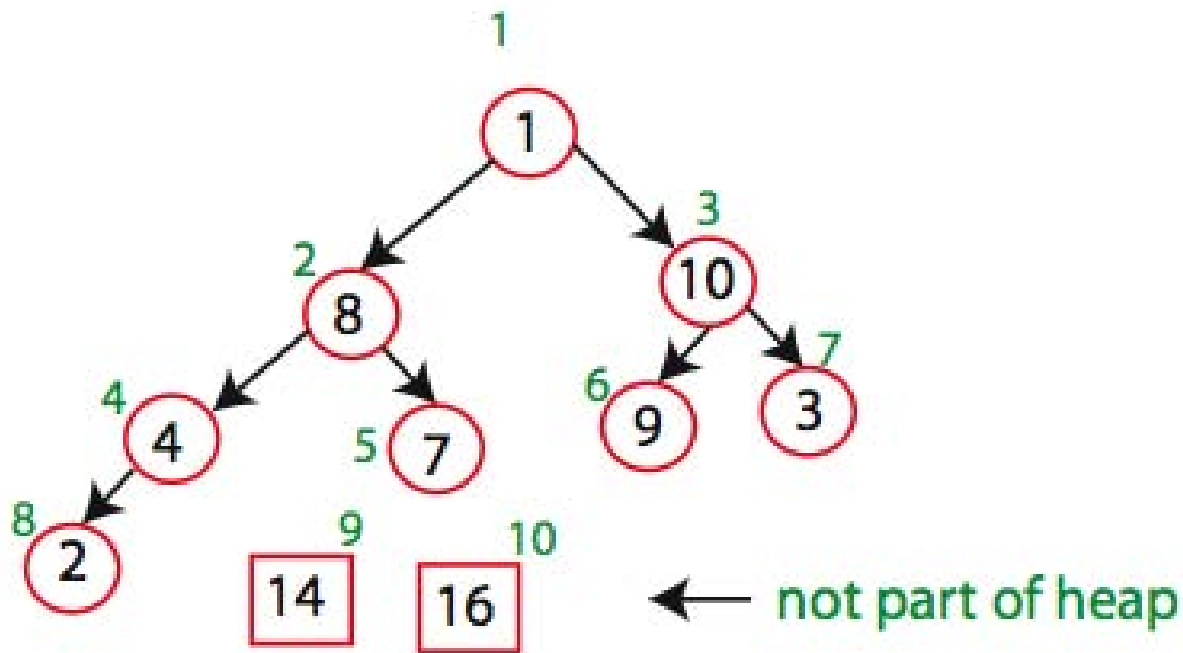


Swap $A[9]$ and $A[1]$

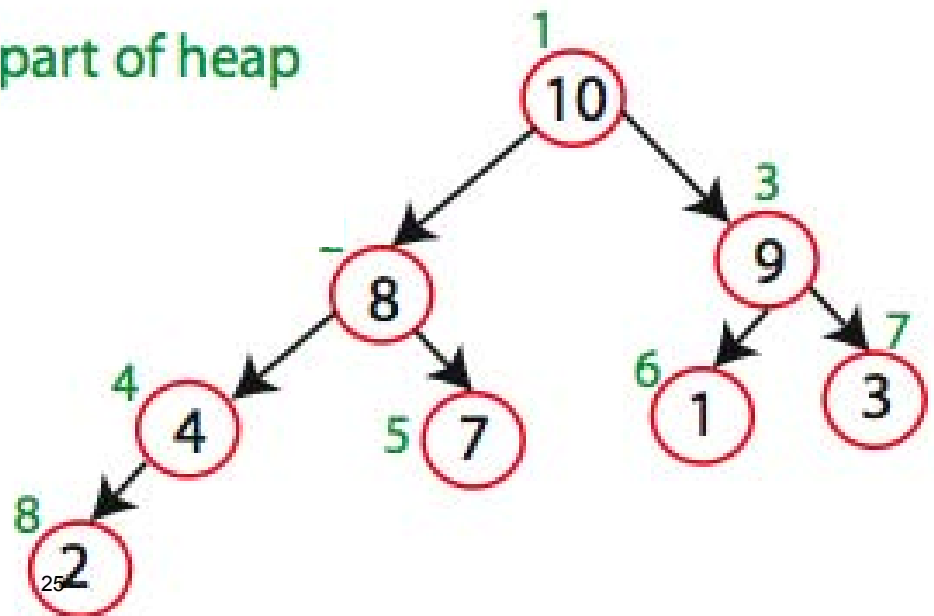


²⁴
MAX_HEAPIFY (A,1)

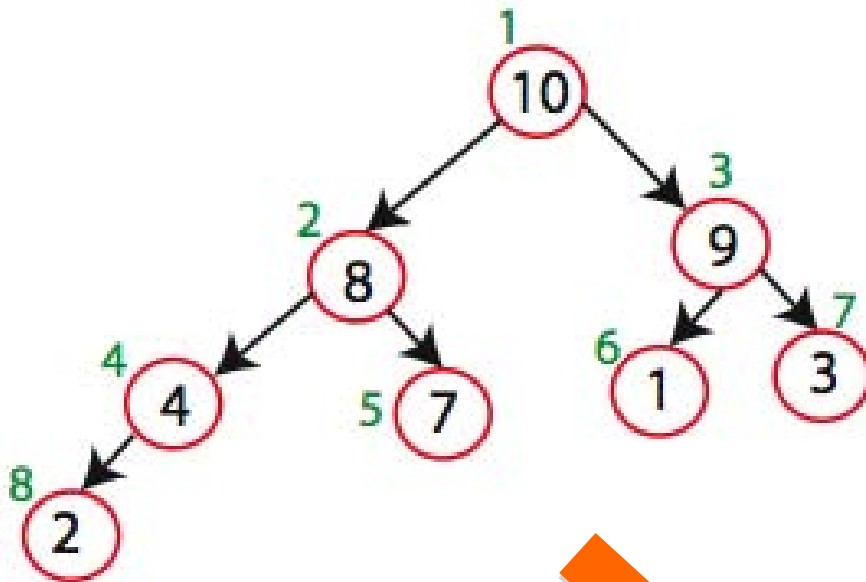
Heap-Sort Demo



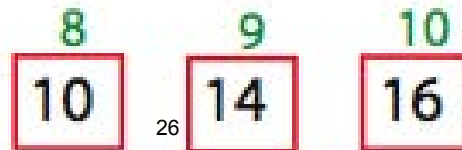
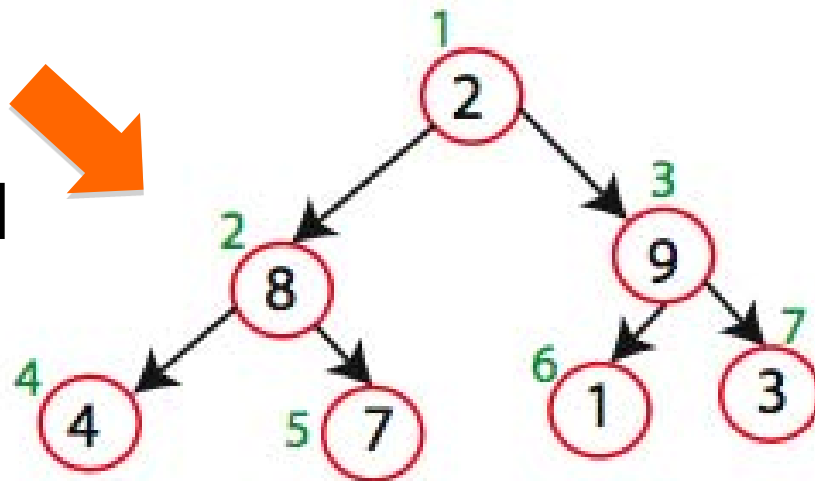
MAX_HEAPIFY (A,1)



Heap-Sort Demo



Swap $A[8]$ and $A[1]$



← not part of heap

Heap-Sort

Running time:

after n iterations the Heap is empty
every iteration involves a swap and a max_heapify
operation; hence it takes $O(\log n)$ time

Overall $O(n \log n)$