Αλγόριθμοι

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The Akra-Bazzi theorem

Introduction

MergeSort is an example of a divide-and-conquer algorithm:

- it divides the input,
- "conquers" the pieces, and
- combines the results
- Analysis of such algorithms commonly leads to *divide-and-conquer recurrences*, which have this form:

$$T(n) = \sum_{i=1}^{k} a_i T(b_i n) + g(n)$$

The Akra-Bazzi formula

The solution to virtually all divide and conquer solutions is given by the amazing *Akra-Bazzi formula*

Quite simply, the asymptotic solution to the general divide-and-conquer recurrence:

$$T(n) = \sum_{i=1}^{k} a_i T(b_i n) + g(n)$$

is

$$T(n) = \Theta\left(n^p\left(1 + \int_1^n \frac{g(u)}{u^{p+1}}\,du\right)\right)$$

where

$$\sum_{i=1}^{k} a_i b_i^p = 1$$

The Akra-Bazzi formula for MergeSort

First, we find the value p that satisfies 2 x (1/2)^p = 1

Looks like p = 1 does the job

Then we compute the integral:

$$
\begin{aligned}
T(n) &= \Theta\left(n\left(1+\int_1^n \frac{u-1}{u^2}du\right)\right) \\
&= \Theta\left(n\left(1+\left[log(u)+\frac{1}{u}\right]_1^n\right)\right) \\
&= \Theta\left(n\left(log(n)+\frac{1}{n}\right)\right) \\
&= \Theta\left(n \times log(n)\right)
\end{aligned}
$$

The Akra-Bazzi formula for a scary case

Let's try a scary-looking recurrence:

$$T(n) = 2T(n/2) + (8/9)T(3n/4) + n^2$$

Here, $a_1=2$, $b_1=1/2$, $a_2=8/9$, $b_2=3/4$

So we find the value of p that satisfies:

$$2(1/2)^p + (8/9)(3/4)^p = 1$$

Equations of this form don't always have closed-form solutions, so you may need to approximate p numerically sometimes

The Akra-Bazzi formula for a scary case

But in this case the solution is simple: p = 2

Then we integrate:

$$T(n) = \Theta\left(n^2\left(1 + \int_1^n \frac{u^2}{u^3}du\right)\right)$$

$$= \Theta\left(n^2\left(1 + log(n)\right)\right)$$

$$= \Theta\left(n^2 \times log(n)\right)$$

The Akra-Bazzi theorem

Theorem. Suppose that the function $T : R \to R$ is nonnegative and bounded for $0 \leq x \leq x_0$ and satisfies the recurrence

$$T(x) = \sum_{i=1}^{k} a_i T(b_i x + h_i(x)) + g(x), \quad \text{for } x > x_0$$

where:

- *x_0 is large enough so that T is well-defined,*
- *$a_1, \ldots, a_k$ are positive constants,*
- *$b_1, \ldots, b_k$ are constants between 0 and 1,*
- *$g(x)$ is a nonnegative function such that $|g'(x)|$ is bounded by a polynomial,*
- $|h_i(x)| = O(x/\log^2(x))$

The Akra-Bazzi theorem

Theorem (cont'ed). Then

$$T(x) = \Theta\left(x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du \right) \right)$$

where p satisfies:

$$\sum_{i=1}^{k} a_i b_i^p = 1$$

The Master theorem

Master Theorem. Let T be a recurrence of the form

$$T(n) = aT(\frac{n}{b}) + g(n)$$

Case 1: if, for some constant ε>0, $g(n) = O(n^{log_b(a)-\epsilon})$

$$T(n) = \Theta\left(n^{log_b(a)}\right)$$

Case 2: if, for some constant k≥0, $g(n) = \Theta(n^{log_b(a)} log^k(n))$

$$T(n) = \Theta\left(n^{log_b(a)} log^{k+1}(n)\right)$$

Case 3: if, for some constant ε>0 and ag(n/b)<cg(n) for some constant c<1 and sufficiently large n, $g(n) = \Omega(n^{log_b(a)+\epsilon})$

$$T(n) = \Theta\left(g(n)\right)$$