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PERFORMANCE EVALUATION OF CONCURRENT SYSTEMS USING PETRI NETS

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Introduction

Timed Petri nets make possible the performance evaluation of concurrent systems. Two approaches to modeling time of operations exist. The first method associates time to transitions [4,9, 10,13], whereas the second one associates it to places [11]. An analysis of their equivalence is given in [12]. Principles of transforming a net with time associated to transitions into a net with time associated to places and conversely are also given in [12]. In the literature the first approach prevails because of the following fact. The General Net Theory [3] is founded upon a system model called condition-event net. In this model conditions are represented by places and events are represented by transitions. Many authors define time by events [1]. Therefore the method with time associated to transitions seems to be more natural and is consequently chosen in this paper.

The literature on performance evaluation by analytical means of the General Net Theory is very scarce. However, existing papers contain apparent important differences and one of our purposes is to indicate interrelations between those papers.

As a performance evaluation measure the cycle time is chosen. This measure enables us to estimate such computer system parameters as system throughput, reaction time, etc.

Basic definitions are given in Section 1. In Section 2, a polynomial algorithm for the minimal cycle time problem for timed marked graphs is sketched. The minimal cycle time problem for timed Petri nets is NP-complete. So algorithms generating exact solutions of this problem most probably must have exponential complexity. Hence, polynomial bounds for the P-invariant Petri nets and non-P-invariant ones are given in Section 3.

1. Definitions

Definition 1. A *Petri net* (PN) is a 5-tuple $\mathcal{N}= \langle P, T, \mathcal{L}, \beta, M_0 \rangle$ where
- P is a set of places,
- T is a set of transitions ($P \cap T = \emptyset$),
- $\mathcal{L}: P \times T \to \mathbb{N}$ is the forward incidence function ($\mathbb{N} = \{0, 1, \dots\}$),
- $\beta : P \times T \to \mathbb{N}$ is the backward incidence function,
- $M_0 : P \to \mathbb{N}$ is the initial marking.

Definition 2. A *timed Petri net* (TPN) is a 6-tuple $\mathcal{N}= \langle P, T, \mathcal{L}, \beta, M_0, \mathcal{T} \rangle$ where P, T, $\mathcal{L}$, β, M_0 are determined by Definition 1, $\mathcal{T}: T \to \mathbb{R}_r^+$ is the firing time function ($\mathbb{R}_r^+$ is the set of non-negative rational numbers).

Definition 3. The set

$$^{\bullet}t = \{p \in P : \mathcal{L}(p, t) \neq 0\}$$

$$(t^{\bullet} = \{p \in P : \beta(p, t) \neq 0\})$$

is called *set of input* (resp. output) *places* of the transition t. The set

$$^{\cdot}p = \{t \in T: \beta(p, t) \neq 0\}$$
$$(p^{\cdot} = \{t \in T: \mathscr{L}(p, t) \neq 0\})$$

is called *set of input* (resp. output) *transitions* of the place p.

Tokens of a TPN have two possible states: reserved and non-reserved one. Only non-reserved tokens can be used in order to enable a transition. If a transition t is enabled, then it can fire by reserving $\mathscr{L}(p, t)$ non-reserved tokens at each input place $p \in {}^{\cdot}t$ during the firing time $\mathscr{T}(t)$. Then $\mathscr{L}(p, t)$ reserved tokens are removed from the places $p \in {}^{\cdot}t$ and $\beta(p, t)$ non-reserved tokens are added to the places $p \in t^{\cdot}$.

Definition 4. An *ordinary* PN is a PN such that

$$\mathscr{L}: P \times T \to \{0, 1\}, \qquad \beta: P \times T \to \{0, 1\}.$$

Definition 5. A *marked graph* (MG) is an ordinary PN such that

$$|{}^{\cdot}p| = |p^{\cdot}| = 1 \quad \forall p \in P.$$

Definition 6. A *pure* PN is a PN such that

$$^{\cdot}t \cap t^{\cdot} = \emptyset \quad \forall t \in T.$$

Definition 7. For a pure PN with $|P| = n$, $|T| = m$, we define the matrices

$$C = [c_{ij}]_{n \times m},$$

$$c_{ij} = \begin{cases} \beta(p_i, t_j) & \text{if } \beta(p_i, t_j) \neq 0, \\ -\mathscr{L}(p_i, t_j) & \text{if } \mathscr{L}(p_i, t_j) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$C^- = [c_{ij}^-]_{n \times m},$$

$$c_{ij}^- = \begin{cases} \mathscr{L}(p_i, t_j) & \text{if } \mathscr{L}(p_i, t_j) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

C is called the *incidence matrix* of the PN.

Hence, a pure PN can be expressed by the tuple $\mathscr{N} = \langle C, M_0 \rangle$.

Throughout this paper we examine live and bounded Petri nets.

Definition 8. A PN is said to be a T-*invariant* (resp. P-invariant) PN iff there exists a column vector I (resp. J) with all its components strictly positive such that $C \cdot I = 0$ (resp. $J^T \cdot C = 0$). I (resp. J) is called T-invariant (resp. P-invariant). I_i (resp. J_i) denotes the *ith* component of a T-invariant I (rsp. P-invariant J).

Let us now explain the above definition.

Let M, M′ be markings. For the sequence, $\rho = t_a \cdots t_i \cdots$, let $\nu(\rho)$ be a firing vector, whose *ith* component is the number of times that the transition t_i fires in the sequence ρ. For the T-invariant I the following holds:

$$M' = M + C \cdot I = M.$$

Hence, the T-invariant is equal to the firing vector for a sequence after which the same marking as before is achieved. Thus the T-invariant can be used to describe cyclic performance. If PN is live and bounded, then PN is a T-invariant PN.

The *ith* component of a P-invariant J determines the weight of a token in the place p_i. The P-invariant satisfies the following:

$$J^T \cdot M' = J^T \cdot M \quad \forall M' \in \vec{M},$$

where $\vec{M}$ is the set of markings reachable from the marking M. Hence, the weighted sum of tokens in places of P-invariant PN remains the same regardless of the firing sequence.

For an MG the space of solutions of $C \cdot I = 0$ is of dimension 1 and the vector $I^T = [11 \cdots 1]$ is a base of this space. For a PN the space of solutions of $C \cdot I = 0$ contains $k = m - \text{rank}[C]$ linearly independent vectors, where rank[C] denotes the rank of the matrix C. Thus each two linearly independent solutions of $C \cdot I = 0$ describe different working modes of the system expressed by the given PN. So we define cycle time for given T-invariant I. During cycle time a transition t_i fires I_i times.

Definition 9. A *time firing sequence* is a sequence

$$x = (t_a, t_a(1))(t_i, t_i(1)) \ldots (t_j, t_j(k)) \ldots (t_i, t_k(m)) \ldots$$

where

– $\rho = t_a t_i \cdots t_j \cdots t_i \cdots$ is the firing sequence,

– $t_i(m)$ is the time at which transition t_i initiates its m*th* firing.

Definition 10. A number γ is called a *cycle time* of a PN for a T-invariant I iff there is a time firing sequence x such that

$$\gamma = \lim_{n \to \infty} \frac{t_i(n \cdot I_i)}{n} \quad \forall t_i \in T$$

where $t_i(n \cdot I_i)$ is the time at which transition t_i initiates its $(n \cdot I_i)$*th* firing.

Definition 11. In a PN, a directed path $p_i t_j p_k \cdots t_p p_r$ such that vertices are different except for places p_i, p_r $(p_i = p_r)$ is called *circuit*.

Definition 12. A PN is *strongly connected* iff for every pair of vertices v_i, $v_r \in P \cup T$ there exists a path directed from vertex v_i to v_r.

In this paper we analyse strongly connected Petri nets. The obtained results can easily be generalized to non-strongly connected Petri nets.

A linear programming problem is formulated as follows:

$$\min c_1 \cdot x_1 + \cdots + c_j \cdot x_j + \cdots + c_n \cdot x_n,$$

$$b_i \leqslant a_{i1} \cdot x_1 + \cdots + a_{ij} \cdot x_j + \cdots + a_{in} \cdot x_n,$$

$$i = 1, \ldots, m,$$

where a_{ij}, b_i, c_j are rational numbers.

The notions of polynomial algorithm, NP-complete problem are used as in [2,5].

2. Marked graphs

An algorithm for verifying timed marked graph performance is given in [10]. The above algorithm makes it possible to verify if performance requirements expressed by a given cycle time can be satisfied. The algorithm can be executed in $O(n^3)$ steps.

We are interested in determining the minimal cycle time. For timed marked graphs we search minimal cycle time for T-invariant $I^T = [11 \cdots 1]$.

Theorem 1 [10]. *For a timed marked graph* (TMG) *the minimal cycle time is given by*

$$\gamma_{\min} = \max_{C_j \in C^*} \frac{T_j}{K_j},$$

such that

$$t_i(n_i) = \tau_i + (n_i - 1) \cdot \gamma_{\min},$$

where

- $T_j = \Sigma_{t_i \in T(C_j)} \mathscr{T}(t_i)$ *is the sum of firing times of the transitions in circuit* C_j,
- $K_j = \Sigma_{p_i \in P(C_j)} M_0(p_i)$ *is the number of tokens in the places in circuit* C_j,
- C^* *is the set of all circuits in* TMG,
- $t_i(n_i)$ *is the time at which transition* t_i *initiates its* n_i*th firing*,
- τ_i *is the time at which transition* t_i *initiates its first firing.*

The proof of the above theorem contains a method of calculating the number τ_i.

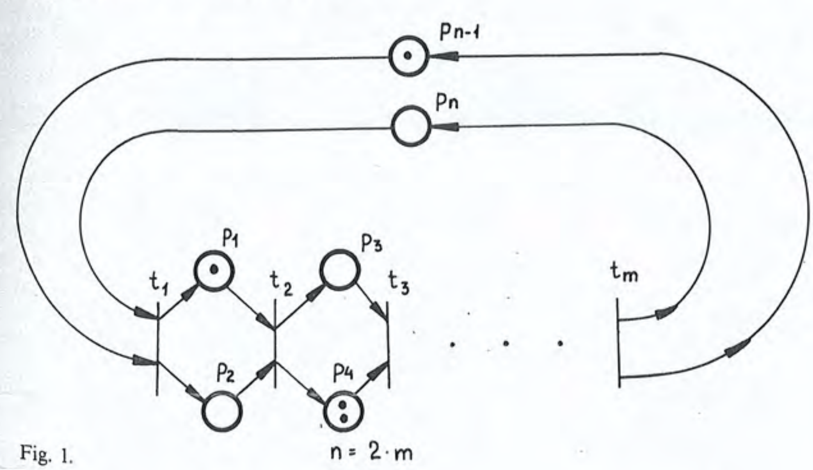

Fig. 1.

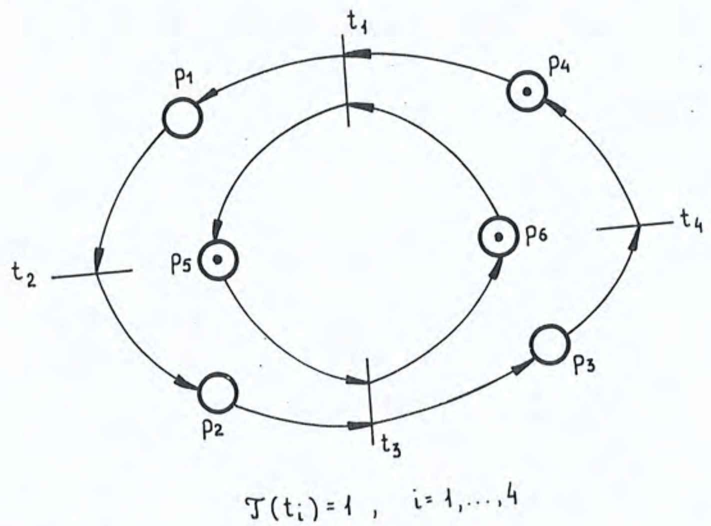

Fig. 2.

There exist marked graphs with an exponential number of circuits.

Example 1. The MG of Fig. 1 contains 2^m circuits. Therefore, the method based on Theorem 1 requires the enumerating of an exponential number of circuits.

In [11], in order to determine the minimal cycle time of an MG with time associated to the places, only the elements of a circuit base are considered. The circuit base of an MG consists of $(n - m + 1)$ elements $(n = |P|, m = |T|)$. As a result of transforming a TMG into an MG with time associated to places [12], the number of base circuits remains the same. Therefore, the above method is polynomial. However, by using the method we can only obtain a lower bound of minimal cycle time.

Example 2. Let us consider TMG of Fig. 2. There exists a certain circuit base C^b which consists of the following circuits:

$$C_1 = p_5 t_3 p_6 t_1 p_5,$$
$$C_2 = p_5 t_3 p_3 t_4 p_4 t_1 p_5,$$
$$C_3 = p_1 t_2 p_2 t_3 p_6 t_1 p_1.$$

Hence

$$\gamma' = \max_{C_i \in C^b} \frac{T_i}{K_i} = \max\left\{\tfrac{2}{2}, \tfrac{3}{2}, \tfrac{3}{1}\right\} = 3.$$

However, for the circuit $C_4 = p_1 t_2 p_2 t_3 p_3 t_4 p_4 t_1 p_1$ we have $T_4/K_4 = \frac{4}{1} = 4$. Thus γ' is a lower bound for $\gamma_{min} = 4$ only.

Let us present now a polynomial method for determining the minimal cycle time of a TMG.

Let us consider a part of a TMG given in Fig. 3. For a cycle time γ the following relations must be satisfied:

termination time of the $n_i th$ firing of transition $t_i \leqslant$ initiation time of the $(n_i + M_0(p_k))th$ firing of transition t_j,

$$t_i(n_i) + \mathscr{T}(t_i) \leqslant t_j(n_i + M_0(p_k)),$$
$$\tau_i + (n_i - 1)\cdot\gamma + \mathscr{T}(t_i) \leqslant \tau_j + (n_i - 1 + M_0(p_k))\cdot\gamma,$$
$$\mathscr{T}(t_i) \leqslant \tau_j - \tau_i + M_0(p_k)\cdot\gamma.$$

To calculate the minimal cycle time, we formulate a linear programming problem

$$\min \gamma,$$
$$\mathscr{T}(t_i) \leqslant \tau_j - \tau_i + M_0(p_k)\cdot\gamma \quad \text{for each } p_k \in P,$$
$$0 \leqslant \gamma,$$

where $\tau_1, \ldots, \tau_{|T|}, \gamma$ are variables.

A polynomial algorithm in the linear programming is given in [6].

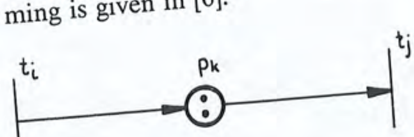

Fig. 3.

Theorem 2. *There exists an polynomial algorithm for the minimal cycle time problem for a* TMG.

The above theorem is mainly of theoretical importance, because the simplex method requires a lower number of operations in many practical cases. However, for the simplex method there are cases where an exponential number of iterations is needed.

3. Petri nets

We consider pure Petri nets only.

It is shown that the minimal cycle time problem for a P-invariant TPN is NP-complete [10]. Bounds for the minimal cycle time problem of a P-invariant TPN are given in [10].

3.1. P-invariant Petri nets

To begin with we present the lower bound for the minimal cycle time proposed in [10]. Subsequently, we formulate a new lower bound and compare them.

The lower bound of the minimal cycle time for a T-invariant I using an S-invariant J is given by

$$\gamma \geqslant \frac{\sum_{t_j \in T}\left(\sum_{p_i \in {}^{\cdot}t_j} J_i \cdot c_{ij}^-\right) \cdot I_j \cdot \mathscr{T}(t_j)}{\sum_{t_j \in T}\left(\sum_{p_i \in {}^{\cdot}t_j} J_i\right)\left(\sum_{p_i \in {}^{\cdot}t_j} M_0(p_i)\right)}. \qquad \text{(LB1)}$$

The bound (LB1) is a generalization of a lower bound given in [10]. The lower bound for an ordinary PN is presented in [10] only.

Let us determine a new lower bound for minimal cycle time. To compute a lower bound for cycle time we can use results of [12]. So we can transform a TPN into a PN with time associated to places. Subsequently, we can apply to the obtained PN results of [11]. However, by using this transformation, the addition of m places and m transitions is required.

For P-invariant Petri nets the following holds:

$$J^T \cdot M = J^T \cdot M_0 \quad \forall M \in \vec{M}_0.$$

Let $M_{i_1}, M_{i_2}, \ldots, M_{i_p}$ be the markings which are successively reached by a P-invariant TPN during the cycle and let $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_p}$ be their respective durations. Then, the mean marking M^* and the P-invariant J satisfy the following:

$$\begin{aligned} J^T \cdot M^* &= J^T \cdot \sum_{j=1}^{p} M_{i_j} \cdot \pi_{i_j} \Big/ \sum_{j=1}^{p} \pi_{i_j} \\ &= \sum_{j=1}^{p} J^T \cdot M_{i_j} \cdot \pi_{i_j} \Big/ \sum_{j=1}^{p} \pi_{i_j} \\ &= \sum_{j=1}^{p} J^T \cdot M_0 \cdot \pi_{i_j} \Big/ \sum_{j=1}^{p} \pi_{i_j} \\ &= J^T \cdot M_0. \end{aligned}$$

The product $c_{ij}^- \cdot I_j$ represents the number of tokens removed from the place p_i as a result of I_j firings of transition t_j during the cycle. The sum of the stay times of $c_{ij}^- \cdot I_j$ tokens in the place p_i is at least equal to $c_{ij}^- \cdot I_j \cdot \mathscr{T}(t_j)$. The sum of the stay times of all tokens in the place p_i during the cycle is at least equal to

$$\sum_{t_j \in p_i^{\cdot}} c_{ij}^- \cdot I_j \cdot \mathscr{T}(t_j).$$

The value $\gamma \cdot M^*(p_i)$ expresses the stay time of the tokens in the place p_i during the cycle. Therefore, γ must satisfy the relation

$$\gamma \cdot M^*(p_i) \geqslant \sum_{t_j \in p_i^{\cdot}} c_{ij}^- \cdot I_j \cdot \mathscr{T}(t_j).$$

Using a P-invariant J we obtain

$$\gamma \geqslant \frac{\sum_{p_i \in P} J_i \cdot \left(\sum_{t_j \in p_i^{\cdot}} c_{ij}^- \cdot I_j \cdot \mathscr{T}(t_j)\right)}{\sum_{p_i \in P} J_i \cdot M_0(p_i)}. \qquad \text{(LB2)}$$

Let us compare the lower bounds (LB1) and (LB2). Their numerators are equal. However, for the denominators D_1 and D_2 the following holds:

$$\begin{aligned} D_1 &= \sum_{t_j \in T}\left(\sum_{p_i \in {}^{\cdot}t_j} J_i\right) \cdot \left(\sum_{p_i \in {}^{\cdot}t_j} M_0(p_i)\right) \\ &= \sum_{p_i \in P} J_i \cdot M_0(p_i) \\ &\quad + \sum_{t_j \in T}\left(\sum_{p_i \in {}^{\cdot}t_j} J_i\right)\left(\sum_{\substack{p_k \in {}^{\cdot}t_j \\ p_k \neq p_i}} M_0(p_k)\right), \end{aligned}$$

where

$$\sum_{p_i \in P} J_i \cdot M_0(p_i) = D_2.$$

Theorem 3. *The lower bound* (LB2) *of minimal cycle time is better than the lower bound* (LB1).

To obtain a better lower bound we can decompose a P-invariant net into a set of P-invariant subnets. Decomposition methods are presented in [7,11]. For every P-invariant subnet we can compute the lower bound (LB2). From the resultant set of bounds we choose the maximal one.

To obtain an upper bound we can choose a sequence which satisfies the T-invariant requirements. Here we can use the results of [13]. In [13] a subset of the set of firing sequences satisfying T-invariant requirements is considered only. Therefore, we do not always obtain the exact value of the minimal cycle time.

3.2. Non-P-invariant Petri nets

To the best of our knowledge, there does not exist a paper covering lower bounds for non-P-invariant Petri nets.

For an non-P-invariant Petri net $\mathcal{N}$ we construct a P-invariant net $\mathcal{N}^{\cdot}$ with identical minimal cycle time. To do so, we add the place p_{n+1} to the net $\mathcal{N}$. The components of the incidence matrix $C^{\cdot}$ for the place p_{n+1} are given by

$$c^{\cdot}_{n+1,j} = -\sum_{i=1}^{n} c_{ij}, \quad j = 1,\ldots,m.$$

For the resultant net $\mathcal{N}^{\cdot}$ there exists the P-invariant $J^{\cdot T} = [1, 1,\ldots,1]$. An initial marking $M^{\cdot}_0$ of the net $\mathcal{N}^{\cdot}$ satisfies the following:

$$M^{\cdot}_0(p_i) = M_0(p_i) \quad \forall p_i \in P,$$

$$M^{\cdot}_0(p_{n+1}) = \sum_{t_j \in T} c^{\cdot -}_{n+1,j} \cdot I_j,$$

where $c^{\cdot -}_{n+1,j}$ is determined by Definition 7.

The subnet $\mathcal{N}$ of the net $\mathcal{N}^{\cdot}$ is a T-invariant net. Hence, after any firing sequence satisfying the requirements of a T-invariant I, the marking of the places of subnet $\mathcal{N}$ is reproduced. Because of this and since the net $\mathcal{N}^{\cdot}$ is P-invariant, the marking of the place p_{n+1} is also reproduced. Consequently vector I is a T-invariant for the net $\mathcal{N}^{\cdot}$.

The initial marking of the place p_{n+1} guarantees that a sequence ρ described by T-invariant I is the firing sequence of the TPN $\mathcal{N} = \langle C, M_0, \mathcal{T} \rangle$ iff ρ is the firing sequence of the TPN $\mathcal{N}^{\cdot} = \langle C^{\cdot}, M^{\cdot}_0, \mathcal{T} \rangle$. Therefore, the minimal cycle times of nets $\mathcal{N}$, $\mathcal{N}^{\cdot}$ for the vector I are equal.

Theorem 4. *For a non-*P*-invariant* TPN *there exists a* P*-invariant* TPN *with identical minimal cycle time for a* T*-invariant* I.

Hence, we can apply the lower bound (LB2) to the net $\mathcal{N}^{\cdot}$.

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