Minimal Deadlocks in Free Choice Nets

Javier Esparza[1], Eike Best[2], Manuel Silva[3]

July 10, 1989

ABSTRACT:

A characterisation of minimal deadlocks in strongly connected free choice Petri nets is derived. An algorithm that constructs minimal deadlocks is given. A close relationship between minimal deadlocks and strongly connected deadlocks is then obtained.

1 Introduction

Deadlocks and traps [4] are very useful structures for the analysis of Petri nets, in particular the so-called minimal ones. In [1] K. Barkaoui and B. Lemaire give a nice characterisation of minimal deadlocks in terms of graph theoretical properties, using the notion of alternating circuit. The purpose of this note is to show that this characterisation reduces to a simpler one for the particular case of free choice nets; then this simplified characterisation is used to obtain two results: the first is an algorithm that constructs minimal deadlocks in (strongly connected) free choice nets; the second is the existence of a close relationship between minimal and strongly connected deadlocks in the same subclass.

This last result has an important consequence: minimal deadlocks in free choice nets enjoy some properties if and only if the strongly connected deadlocks enjoy them (one of these properties, as will be shown at the end of the note, is that of being a trap). As strongly connected deadlocks are easier to handle, this reduces the complexity of dediciing these properties. We will also show that this relationship allows to characterise liveness of bounded free choice nets in terms of strongly connected deadlocks. The note is organised as follows. Basic definitions are given in section 1. Section 2 contains the simplified characterisation. The results derived from it appear in section 3.

[1]Depto. de Ing. Eléctrica e Informática, Universidad de Zaragoza

[2]Institut für Informatik, Universität Hildesheim

[3]Depto. de Ing. Eléctrica e Informática, Universidad de Zaragoza

2 Definitions and basic results

Definition 2.1 *Basic definitions*

A *place/transition net* (or *P/T net*) is a triple $N = (S, T, F)$ where

- S is the set of *places*,
- T is the set of *transitions* ($S \cap T = \emptyset$),
- $F \subseteq (S \times T) \cup (T \times S)$ is the *flow relation.*

The elements of $S \cup T$ are called *nodes*. The *Pre-set* ${}^\bullet x$ of $x \in S \cup T$ is given by ${}^\bullet x = \{y \in S \cup T \mid (y, x) \in F\}$. The *Post-set* $x^\bullet$ of $x \in S \cup T$ is given by $x^\bullet = \{y \in S \cup T \mid (x, y) \in F\}$. The Pre-set or Post-set of a set of nodes is the union of the Pre-sets or Post-sets of its elements. A function $M: S \to \mathbf{N}$ is called a *marking*. A *marked P/T net*, $< N, M_0 >$, is a P/T net, N, with an initial marking M_0. A transition $t \in T$ is *enabled at* M iff $\forall p \in {}^\bullet t: M(p) > 0$. If $t \in T$ is enabled at a marking M then t may *occur* yielding a new marking M' given by $M'(p) = M(p) - \overline{F}(p, t) + \overline{F}(t, p)$ for all $p \in S$ (where $\overline{F}$ is the characteristic function of F). $M[t\rangle M'$ denotes the fact that M' is reached from M by the occurrence of t.
A sequence of transitions, $\sigma = t_1 t_2 \dots t_n$, is a *transition sequence of* (N, M_0) iff there exists a sequence $M_0 t_1 M_1 t_2 M_2 \dots t_n M_n$ such that

$$\forall i, 1 \leq i \leq n: M_{i-1}[t_i\rangle M_i.$$

The marking M_n is said to be *reachable* from M_0 by the occurrence of σ. The set of markings reachable from M_0 is denoted by $R(N, M_0)$. ■ 2.1

Definition 2.2 *Net subclasses*

A net $N = (S, T, F)$ is called an *S–graph* iff each transition has exactly one input place and one output place, i.e. $\forall t \in T: |{}^\bullet t| = 1 = |t^\bullet|$. N is called a *T–graph* iff each place has exactly one input transition and one output transition, i.e. $\forall s \in S: |{}^\bullet s| = 1 = |s^\bullet|$. N is called *free choice* iff for all $p \in S$ such that $|p^\bullet| > 1$, we have ${}^\bullet(p^\bullet) = \{p\}$. ■ 2.2

Definition 2.3 *Subnets*

$N' = (S', T', F')$ is a *subnet* of $N = (S, T, F)$ iff $S' \subseteq S$, $T' \subseteq T$ and $F' = F \cap ((S' \times T') \cup (T' \times S'))$. N' is a *partial subnet* of N iff $S' \subseteq S$, $T' \subseteq T$ and $F' \subseteq F \cap ((S' \times T') \cup (T' \times S'))$. N' is an *S–component (T–component)* of N iff N' is a strongly connected S–graph (T–graph) and $T' = {}^\bullet S' \cap S'^\bullet$ ($S = {}^\bullet T' \cup T'^\bullet$). N' is *generated* by a set $X' \subseteq S \cap T$ iff

$$\begin{aligned} S' &= (X' \cap S) \cup {}^\bullet(X' \cap T) \cup (X' \cap T)^\bullet \\ T' &= (X' \cap T) \cup {}^\bullet(X' \cap S) \cup (X' \cap S)^\bullet. \end{aligned}$$

■ 2.3

Definition 2.4 *Behavioural properties*

A marked net (N, M_0) is said to be *bounded* iff

$$\forall p \in S\ \exists k \in \mathbf{N}\ \forall M \in R(N, M_0) \colon M(p) \leq k.$$

A net N is *structurally bounded* iff it is bounded for any initial marking M_0. A transition $t \in T$ is *live* in (N, M_0) iff

$$\forall M \in R(N, M_0)\ \exists M' \in R(N, M) \colon M' \text{ enables } t.$$

The marked net (N, M_0) is *live* iff all $t \in T$ are live. N is *structurally live* iff there is some initial marking M_0 such that (N, M_0) is live. ■ 2.4

Definition 2.5 *Graph theoretical concepts*

A *path* of a net $N = (S, T, F)$ is an alternating sequence $\pi = (x_0 f_0 x_1 \dots f_{r-1} x_r)$ of elements of $X = S \cup T$ and F such that

$$\forall i, 0 \leq i \leq r-1 \colon f_i = (x_i, x_{i+1}) \in F.$$

A path is *elementary* iff all x_i are distinct, except possibly x_0 and x_r. A *circuit* is a path such that $x_0 = x_r$. A circuit is *elementary* iff it is elementary as a path. Let N' be a partial subnet of N. An elementary path $\pi = (x_0 f_0 \dots f_{n-1} x_n)$ is a *handle* of N' iff $\pi \cap N' = \{x_0, x_n\}$ (note that if $x_0 = x_n$ then $\{x_0, x_n\} = \{x_0\}$). ■ 2.5

Definition 2.6 *Deadlocks and traps*

Let $N = (S, T, F)$ be a net. $D \subseteq S$ is a *deadlock* iff $D \neq \emptyset$ and ${}^\bullet D \subseteq D^\bullet$. $\Theta \subseteq S$ is a *trap* iff $\Theta \neq \emptyset$ and $\Theta^\bullet \subseteq {}^\bullet\Theta$. A deadlock D is minimal iff there exists no deadlock D' such that $D' \subset D$ (proper inclusion). A deadlock D is *strongly connected* iff the subnet generated by $D \cup {}^\bullet D$ is strongly connected. ■ 2.6

An immediate consequence of the definition of deadlock and trap is the following property:

Proposition 2.7

Let S_1, S_2 be two deadlocks (traps) of N. Then $S_1 \cup S_2$ is also a deadlock (a trap) of N. ■ 2.7

We will make use later of the relationship between minimal and strongly connected deadlocks that is stated below, which holds for any net. It is assumed that a net composed by one place and no transitions is strongly connected.

Proposition 2.8 [8]

Minimal deadlocks are strongly connected. ■ 2.8

The next definition introduces the concept of alternating circuit, upon which Barkaoui and Lemaire base their characterisation of minimality. Although it is necessary to enclose it in order to understand the statement of the subsequent theorem, the concept will be no longer required for the characterisation in free choice nets.

Definition 2.9 [1]

Let $N = (S, T, F)$ be a net. A circuit Γ of N (not necessarily elementary) is an *alternating circuit* iff for all arcs in Γ of the form (p, t), the equality $^\bullet t = \{p\}$ holds. ■ 2.9

3 Characterisation of minimal deadlocks in free choice nets

The characterisation of minimal deadlocks in general Petri nets given by Barkaoui and Lemaire is as follows.

Theorem 3.1 [1]

Let $N = (S, T, F)$ be a net, $D \subseteq S$ a deadlock of N and N_D the subnet of N generated by $D \cup {}^\bullet D$. D is minimal iff there exists a circuit Γ in N_D (not necessarily elementary) that passes through all the places of D such that for every transition $t \in \Gamma$ either:

$|^\bullet t \cap D| = 1$ *or*

$|^\bullet t \cap D| \geq 2$ *and the places of* $(^\bullet t \cap D)$ *belong to an alternating circuit.*

■ 3.1

Now we address the following question: Is there a simpler characterisation for the subclass of free choice nets? The answer is positive, and the reason lies essentially in the following well known lemma.

Lemma 3.2 [6,3]

Let $N = (S, T, F)$ be a free choice net and $D \subseteq S$ a minimal deadlock of N. Then for every $t \in {}^\bullet D$, $|^\bullet t \cap D| = 1$. ■ 3.2

That is, at most one input place of a transition belongs to a given minimal deadlock. It is then clear that the second case of theorem 1 cannot occur in free choice nets. This is what the two following corollaries say, in different forms.

Corollary 3.3

Let $N = (S, T, F)$ be a free choice net, $D \subseteq S$ a deadlock of N and N_D the subnet of N generated by $D \cup {}^\bullet D$. D is minimal iff there exists a circuit Γ in N_D (not necessarily elementary) that passes through all the places of D such that for every transition $t \in \Gamma$ we have $|{}^\bullet t \cap D| = 1$.

Proof: Obvious by applying theorem 3.1 and lemma 3.2. ■ 3.3

Corollary 3.4

Let $N = (S, T, F)$ be a free choice net, $D \subseteq S$ a deadlock of N and N_D the subnet of N generated by $D \cup {}^\bullet D$. D is minimal iff D is strongly connected and for every transition $t \in N_D$ we have $|{}^\bullet t \cap D| = 1$.

Proof: ($\Rightarrow$): If D is minimal, it is strongly connected by proposition 2.8. Every transition $t \in N_D$ satisfies $|{}^\bullet t \cap D| = 1$ by lemma 3.2.

($\Leftarrow$): Since D is strongly connected, N_D is strongly connected. It is then obvious that N_D contains a circuit Γ (possibly non–elementary) that passes through all the places of D. As every transition of N_D has one input place in D, the same happens to the transitions of Γ. ■ 3.4

This last corollary is the characterisation we were looking for. The next section explores its utility.

4 Consequences

We derive from the characterisation given in section 3 a chain of three results. The first is an algorithm that constructs minimal deadlocks in strongly connected free choice nets. This algorithm is then used to show that, in free choice nets, strongly connected deadlocks are the union of minimal deadlocks. Finally, this second result allows us to show that liveness of bounded free choice nets can be characterised in terms of strongly connected deadlocks. The algorithm closely resembles the one given in [2] to construct T-components. We will formalize it in an analogous way.

Algorithm 4.1 *(To construct a minimal deadlock containing a given place)*

Input: A strongly connected free choice net $N = (S, T, F)$ with a distinguished place $\overline{p}$. This place $\overline{p}$ is called the *seed* of the algorithm.

Output: A minimal deadlock of N containing $\overline{p}$.

We construct inductively a net $\overline{N} = (\overline{S} \subseteq S, \overline{T} \subseteq T, \overline{F} \subseteq F)$ such that $\overline{S}$ will turn out to be a minimal deadlock of N. In the following the dot notation $^\bullet$ for Pre– and Post–sets will always refer to the net N.

Step 1: $\overline{S} := \{\overline{p}\}$, $\overline{T} := \emptyset$, $\overline{F} := \emptyset$ and $\overline{N} := (\overline{S}, \overline{T}, \overline{F})$.

Step 2: Repeat the following exhaustively: If there is $p \in \overline{S}$ and $t \in {}^\bullet p$ such that $(t,p) \notin \overline{F}$ then choose a handle $H = (x_0 f_0 x_1 \ldots x_{m-1} f_{m-1} x_m)$ of $\overline{N}$ with $x_{m-1} = t$ and $x_m = p$ (note that $m \geq 1$ and the equality can occur). Then put:

$$\begin{array}{rcl} \overline{S} & := & \overline{S} \cup \{ \text{ places of } H \ \} \\ \overline{T} & := & \overline{T} \cup \{ \text{ transitions of } H \ \} \\ \overline{F} & := & \overline{F} \cup \{ \text{ arcs of } H \ \} \\ \text{and } \overline{N} & := & (\overline{P}, \overline{T}, \overline{F}). \end{array}$$

■ 4.1

Let us now collect five simple properties of the construction. The first three hold at every stage of the algorithm.

1) $\overline{N}$ is a partial subnet of N.

2) $\overline{N}$ is strongly connected in terms of $\overline{F}$.

At the very beginning, $\overline{N}$ is trivially strongly connected and adding handles to it does not destroy the strong connectedness.

3) Every transition in $\overline{T}$ has exactly one incoming $\overline{F}$ arc.

It has at least one because $\overline{N}$ is strongly connected and $\overline{N}$ cannot contain isolated transitions. It has at most one, because this is trivially true at the very beginning and the addition of the particular handles considered in the algorithm does not destroy this property: the new transitions added by the handle have at most one incoming arc because handles are by definition elementary paths. And, as the last node of the handles added to $\overline{N}$ is always a place, no transition already present in $\overline{N}$ can find properly increased its number of incoming arcs by the presence of the new handles.

4) At the end of the algorithm (which clearly terminates, due to the finiteness of N), if $p \in \overline{S}$ then all the incoming arcs of p in F are also in $\overline{F}$ (and therefore, ${}^\bullet p \subseteq \overline{T}$).

The reason is that there always exists, at each stage of the algorithm, at least one handle satisfying the requirements; this follows easily from the strong connectedness of N.

5) At the end of the algorithm $\overline{N}$ is a subnet of N (and, of course, $\overline{N}$ is generated by $\overline{S} \cup \overline{T}$).

Assume the contrary. Then there exists an F–arc f between two nodes of $\overline{N}$ which is not an $\overline{F}$–arc. Two possibilities have to be considered: f leads from a transition to a place or from a place to a transition. The first is easily discarded because it contradicts property 4. Consider the second: if f leads from a place p to a transition t, as $\overline{N}$ is strongly connected it has to be the case that $|p^\bullet| > 1$ and $|{}^\bullet t| > 1$ (recall that the dot notation always refers to N). This is excluded by the free choice property.

Now we are ready to show, using corollary 3.4, that $\overline{S}$ is a minimal deadlock of N, proving thus the correctness of algorithm 4.1.

Theorem 4.2

Let $N = (S, T, F)$ be a strongly connected free choice net, $\overline{p} \in S$ a place of N and $\overline{N} = (\overline{S}, \overline{T}, \overline{F})$ a net constructed using algorithm 4.1 with $\overline{p}$ as seed. Then $\overline{P}$ is a minimal deadlock of N.

Proof: We show first that $\overline{S}$ is a deadlock of N. As $\overline{T} \subseteq \overline{S}^\bullet$ by construction, and $\overline{T} = {}^\bullet\overline{S}$ (property 4), it follows that ${}^\bullet\overline{S} \subseteq \overline{S}^\bullet$. Moreover, $\overline{S}$ is a strongly connected deadlock because $\overline{N}$ is the subnet generated by $\overline{S} \cup \overline{T} = \overline{S} \cup {}^\bullet\overline{S}$ (property 5); $\overline{N}$ is strongly connected (property 2). Finally, every transition $t \in \overline{T}$ satisfies $|{}^\bullet t \cap \overline{P}| = 1$ (property 3). Applying corollary 3.4, $\overline{P}$ is a minimal deadlock of N. ■ 4.2

One interesting remark about algorithm 4.1 is that, when (N, M_0) is a live and bounded free choice net, the net $\overline{N}$ given by the algorithm is an S–component of N. This is so because of Hack's theorem, which states that, in this class of nets, the set of subnets generated by the minimal deadlocks coincides with the set of S–components. Let us now study the relationship between minimal and strongly connected deadlocks. Before obtaining the result announced above, it is convenient to introduce a small lemma.

Lemma 4.3

Let $N = (S, T, F)$ be a net, $S' \subseteq S$ and $N_{S'}$ the subnet generated by $S' \cup {}^\bullet S'$. Then $D' \subseteq S'$ is a deadlock of N iff it is a deadlock of $N_{S'}$.

Proof: From $D' \subseteq S'$ it follows that ${}^\bullet D' \subseteq {}^\bullet S'$. Label this relation as (1).

($\Rightarrow$): Assume D' is a deadlock of N. Then ${}^\bullet D' \subseteq D'^\bullet$. By (1), ${}^\bullet D' \subseteq D'^\bullet \cap {}^\bullet S'$ and therefore ${}^\bullet D \cap {}^\bullet S' \subseteq {}^\bullet D' \subseteq D'^\bullet \cap {}^\bullet S'$. This means that D' is a deadlock of $N_{S'}$.

($\Leftarrow$): Assume D' is a deadlock of $N_{S'}$. Then ${}^\bullet D' \cap {}^\bullet S' \subseteq D'^\bullet \cap {}^\bullet S'$. Using (1), ${}^\bullet D' = {}^\bullet D' \cap {}^\bullet S' \subseteq D'^\bullet \cap {}^\bullet S'$. This implies ${}^\bullet D' \subseteq D'^\bullet$, which means that D' is a deadlock of N. ■ 4.3

Corollary 4.4

Let $N = (S, T, F)$ be a free choice net and $D \subseteq S$ a strongly connected deadlock of N. Then $D = \bigcup D_i$, where the D_i are minimal deadlocks of N.

Proof: Let $N_D = (D, T_D, F_D)$ be the subnet of N generated by $D \cup {}^\bullet D$. N_D is strongly connected by definition and is obviously also free choice. Using algorithm 4.1, given $p \in D$ it is possible to construct a minimal deadlock D_p of N_D containing p. Therefore the claim is true for N_D. It remains to prove that D_p is also a minimal deadlock of N. Using lemma 4.3 with $D = S'$ we obtain that D_p is a deadlock of N. Assume D_p is not minimal in N. Then it contains a minimal deadlock of N, D'. But, again by lemma 4.3, D' is a deadlock of N_D, and as $D' \subseteq D_p$ this contradicts the hypothesis that D_p was a minimal deadlock of N_D. Therefore D_p is a minimal deadlock of N. ■ 4.4

Figure 1 illustrates this result. Consider the net on the left which is not free choice. $D = \{p_1, p_2, p_3\}$ is a strongly connected deadlock. Nevertheless D cannot be covered by minimal deadlocks, because the only minimal deadlock is $\{p_1, p_2\}$. Now add a transition t_5 and a place p_4 to make the net free choice (see the net on the right). $D' = \{p_1, p_2, p_3, p_4\}$ is again a strongly connected deadlock, but now D' can be covered by the minimal deadlocks $\{p_1, p_2, p_4\}$ and $\{p_1, p_2, p_3\}$.

It is true for any net that the union of minimal deadlocks yields a set of strongly connected deadlocks (this follows easily from propositions 2.7 and 2.8). The previous corollary shows that, in the case of free choice nets, a sort of converse of that property holds: every strongly connected deadlock can be decomposed into minimal deadlocks. This fact has a nice consequence, namely that liveness of *bounded* free choice nets can be characterised in terms of strongly connected deadlocks. The proof of this claim makes use of two important results proven in [6].

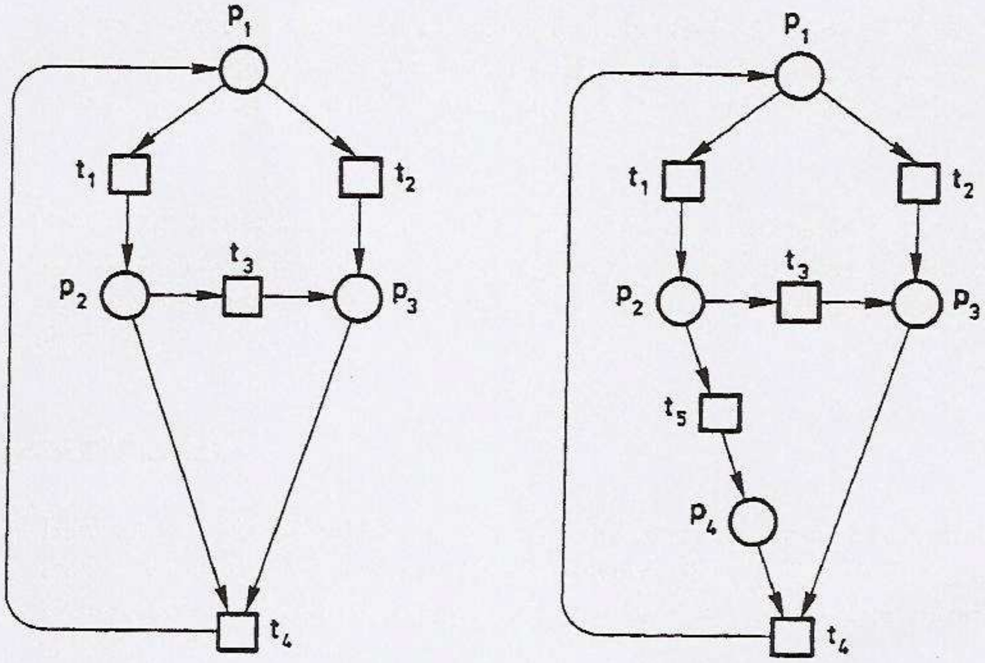

Figure 1: Illustration of corollary 4.4

Theorem 4.5 [6]

Let (N, M_0) be a marked free choice net. (N, M_0) is live iff every minimal deadlock of N contains a marked trap (a trap containing at least one marked place). ■ 4.5

Theorem 4.6 [6,3]

Let (N, M_0) be a live and bounded free choice net and $D \subseteq S$ a minimal deadlock of N. Then D is also a trap. ■ 4.6

Theorem 4.7

Let $(N, M_0) = (S, T, F, M_0)$ be a bounded free choice net. (N, M_0) is live iff every strongly connected deadlock D of N is a marked trap.

Proof: ($\Rightarrow$): By proposition 2.7, if minimal deadlocks are marked traps, their unions are marked traps as well. But by corollary 4.4 the set of these unions contains the set of strongly connected deadlocks.

($\Leftarrow$): Suppose that every strongly connected deadlock D of N is a marked trap. As minimal deadlocks are strongly connected (proposition 2.8), the condition of theorem 4.5 holds and (N, M_0) is live.

■ 4.7

An important remark about theorem 4.7: this new characterisation can be checked in polynomial time [5]. This appears not to be possible for general free choice nets, because the non–liveness problem for free choice nets is NP–complete [7].

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