The prob of bit being zero is $\left(1-\frac{1}{m}\right)^{kn}$

Recall:

$$\underbrace{\underset{\text{prob } 1}{\frac{1}{m}} \rightarrow 1-\underset{\text{prob } \phi}{\frac{1}{m}}}_{\text{after one hash}} \rightarrow \underbrace{\left(1-\frac{1}{m}\right)^{k}}_{\text{after k hash}} \longrightarrow \underbrace{\left(1-\frac{1}{m}\right)^{kn}}_{\text{after all n elements}}$$

The prob of bit being one is $1-\left(1-\frac{1}{m}\right)^{kn}$

To check wheather a given element is in a Bloom Filter

$$f_e^{Bf}(m,k,n) = \left(1-\left(1-\frac{1}{m}\right)^{kn}\right)^{k} \approx \left(1-e^{-kn/m}\right)^{k}$$

$$\frac{\partial f_e^{Bf}(m,t,n)}{\partial k} = \phi$$

$$f_e^{Bf}(m,t,n) = \left(1-e^{-tn/m}\right)^{k} = e^{\ln\left(\left(1-e^{-tn/m}\right)^{k}\right)} = e^{k\ln\left(1-e^{-tn/m}\right)}$$

It is equivalent (and better) to work with g where:

$$g(m,k,n) = k\ln\left(1-e^{-kn/m}\right)$$

$$g' = \frac{\partial g}{\partial k} = \phi \Rightarrow \frac{\partial}{\partial k}\left(k\ln\left(1-e^{-tn/m}\right)\right) = \phi \Rightarrow$$

$$\Rightarrow \ln\left(1-e^{-kn/m}\right) + k\frac{\frac{n}{m}e^{-kn/m}}{1-e^{-kn/m}} = \phi \Rightarrow k = \ln(2)\frac{m}{n}$$

opt k

Considering that: $e^{-kn/m}\Big|_{k=\ln(2)\frac{m}{n}} = e^{-\ln(2)} = \frac{1}{2}$,

then $g'\Big|_{k=\ln(2)\frac{m}{n}} = \ln\left(1-\frac{1}{2}\right)\ln(2)\frac{m}{n}\frac{\frac{n}{2m}}{1-\frac{1}{2}} = \ln\left(\frac{1}{2}\right)\ln(2)\frac{\frac{1}{2}}{\frac{1}{2}} = \ln(1) = \phi$

partial derivative is indeed ϕ

$$f_e^{Bf}(m,k,n)\Big|_{k=\ln(2)\frac{m}{n}} = \left(1-\frac{1}{2}\right)^{\frac{\ln(2)m}{n}} = 0.6185^{\frac{m}{n}}$$

value of f_e^{Bf} for optimal k

Detailed derivation of the formula for the optimal value for k (number of hash functions):

$$\ln\left(1 - e^{\frac{-kn}{m}}\right) + k\frac{\frac{n}{m}e^{\frac{-kn}{m}}}{1 - e^{\frac{-kn}{m}}} = o$$

Let $p = e^{\frac{-kn}{m}} \rightarrow \ln(p) = \frac{-kn}{m}$

The first equation becomes:

$\ln(1-p) - \ln(p)\frac{p}{1-p} = 0 \rightarrow (1-p) \times \ln(1-p) = p \times ln(p) \rightarrow (1-p)^{1-p} = p^p \rightarrow 1 - p = p \rightarrow$
$p = \frac{1}{2}$

Therefore, the second equation gives the following: $ln\frac{1}{2} = \frac{-kn}{m} \rightarrow \frac{kn}{m} = \ln(2) \rightarrow k = \ln(2)\frac{m}{n}$