Κινητός και Διάχυτος Υπολογισμός (Mobile & Pervasive Computing)

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Διάλεξη 21η

The honeycomb conjecture

- **The honeycomb conjecture** (open for about 2000 years)
  - Any partition of the plane into regions of equal area has perimeter at least that of the regular hexagonal honeycomb tiling
    - ✓ The first record of the conjecture dates back to 36 BC, from Marcus Terentius Varro
    - ✓ Pappus of Alexandria (290-350) presented an incomplete proof of the conjecture, based largely on the fact that only three regular polygons (the triangle, the square and the hexagon) fill out a plane
  - Proved by Thomas C. Hale in June 1999
    - ✓ https://link.springer.com/article/10.1007/s004540010071
    - ✓ He is the one who proved **Kepler's conjecture** as well

Kepler's conjecture

- **Kepler's conjecture**
  - It states that no arrangement of equally sized spheres filling space has a greater average density than that of the cubic close packing (face-centered cubic) and hexagonal close packing arrangements. The density of these arrangements is around 74.05%
    - ✓ It was first stated by Johannes Kepler (1611) in his paper 'On the six-cornered snowflake', and mentioned by Hilbert in his famous 1900 problem list
  - Proved by Thomas C. Hale in August 9th, 1998
    - ✓ www.mat.univie.ac.at/~neum/contrib/fullkepler-1.pdf
    - ✓ https://www.cambridge.org/core/journals/forum-of-mathematics-pi/article/formal-proof-of-the-kepler-conjecture/78FBD5E1A3D1BCCB8E0D5B0C463C9FBC
    - ✓ https://link.springer.com/article/10.1007/s00454-009-9148-4

Cannonballs stacked in a face-centred-cubic lattice
(Arlington, Virginia, about 1863)

Κάλυψη και συνδεσμικότητα σε 3D ασύρματα δίκτυα

Motivation

- Conventional network design
  - Almost all wireless terrestrial network based on 2D
  - In cellular system, hexagonal tiling is used to place base station for maximizing coverage with fixed radius
- In Reality: Distributed over a 3D space
  - Length and width are <u>not</u> significantly larger than height
  - Deployed in space, atmosphere or ocean
  - Underwater acoustic ad hoc and sensor networks
  - Army: unmanned aerial vehicles with limited sensing range or underwater autonomous vehicles for surveillance
  - Climate monitoring in ocean and atmosphere

Problem Statement

Assumptions

- All nodes have the same sensing range and same transmission range
- Sensing range $R \leq$ transmission range
- Sensing is omnidirectional, sensing region is sphere of radius R
- Boundary effects are negligible: R<<L, R<<W, R<<H
- Any point in 3D must be covered by (within R of) at least one node
- Free to place a node at any location in the network

Two goals of the work

- 1: Node Placement Strategy) Given R, <u>minimize</u> the number of nodes required for surveillance while guaranteeing 100% coverage. Also, determine the locations of the nodes.
- 2: <u>Minimum</u> ratio) between the transmission range and the sensing range, such that all nodes are connected to their neighbors

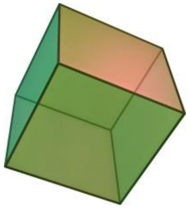
Roadmap

- **Proving optimality in 3D problems**
  - Very difficult, still open for the centuries!
  - E.g., Kepler's conjecture (1611) and proven only in 1998!
  - E.g., Kelvin's conjecture (1887) has not been proven yet!
    - ✓ It is the analogous of Honeycomb conjecture in 3D
- **Instead of proving optimality**
  - Show similarity between our problem and Kelvin's problem.
  - Use Kelvin's conjecture to find an answer to the first question.
  - Any rigorous proof of our conjecture will be very difficult.
  - Instead of giving a proof:
    - ✓ provide detailed comparisons of the suggested solution with thre e other

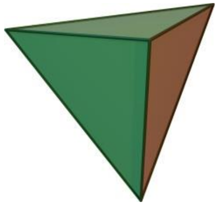
    plausible solutions, and
    - ✓ show that the suggested solution is indeed superior.

Space-Filling Polyhedron

- **Polyhedron**
  - is a 3D shape consisting of a finite number of polygonal faces. E.g., cube, prism, pyramid
- **Space-Filling Polyhedron** (https://mathworld.wolfram.com/Space-FillingPolyhedron.html)
  - is a polyhedron that can be used to fill a volume without any overlap or gap (a.k.a, tessellation or tiling)
  - In general, it is not easy to show that a polyhedron has the space-filling property



Cube (6{4}) is space-filling



In 350 BC, Aristotle claimed that the tetrahedron (4{3}) is space-filling, but his claim was incorrect. The mistake remained unnoticed until the 16th century!

Why Space-Filling?!

- How is our problem related to space-filling polyhedra?
  - Sensing region of a node is spherical
  - Spheres do NOT tessellate in 3D
  - We want to find the space-filling polyhedron that "best approximates" a sphere.

  - Once we know this polyhedron:
    - ✓ Each cell is modeled by that polyhedron (for simplicity), where the distance from the center of a cell to its farthest corner is not greater than R
    - ✓ Number of cells required to cover a volume is minimized
    - ✓ This solves our first problem
  - The question still remains: What is this polyhedron?!

Kelvin's Conjecture

Lord Kelvin (1824 - 1907)

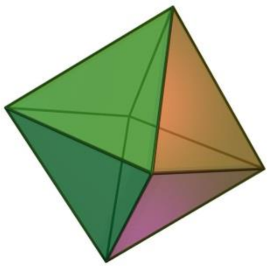
- **In 1887, Lord Kelvin asked:**
  - What is the optimal way to fill a 3D space w cells of equal volume, so that the surface area is minimized?
  - Essentially the problem of finding a space-filling structure having the highest *isoperimetric quotient:*

$$36 \pi V^2 / S^3$$

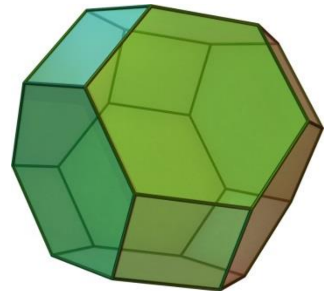
where V is the volume and S is the surface area

    - https://mathworld.wolfram.com/IsoperimetricQuotient.html
  - Sphere has the highest isoperimetric quotient = 1
  - Kelvin's answer: 14-sided truncated octahedron having a very slight curvature of the hexagonal faces and its isoperimetric quotient = 0.757

Truncated Octahedron



Octahedron (8{3}) is
NOT space-filling



Kelvin's tetrakaidecahedron:

Truncated Octahedron (6{4} + 8{6}) is space-filling. The solid of edge length a can be formed from an octahedron of edge length 3a via truncation by removing six square pyramids, each with slant height and base = a

$$S = \left(6 + 12\sqrt{3}\right) a^2.$$

$$V = V_{\text{octahedron}} - 6\,V_{\text{square pyramid}}$$

$$V_{\text{octahedron}} = \frac{1}{3}\sqrt{2}\,(3a)^3 = 9\sqrt{2}\,a^3$$

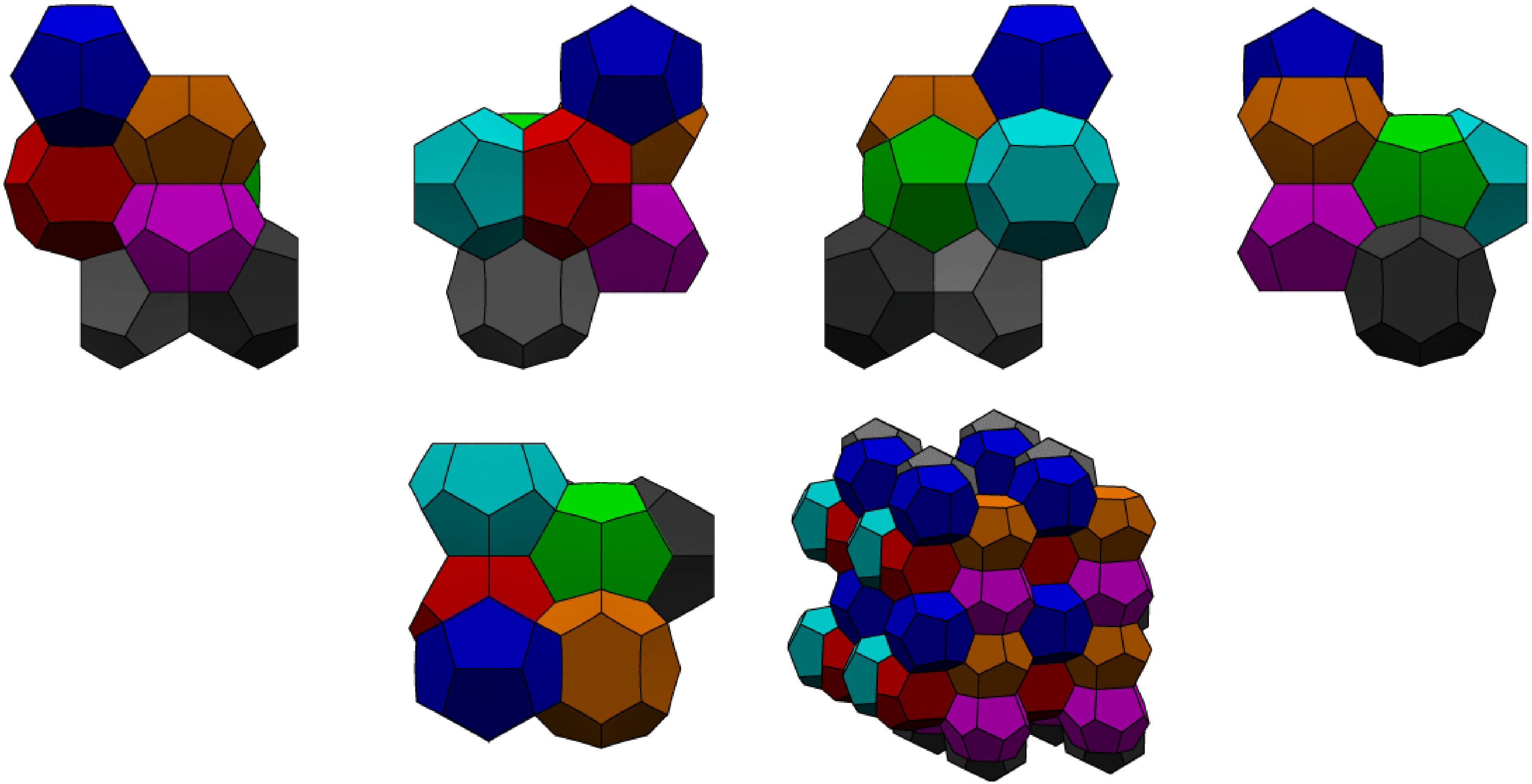
$$V_{\text{square pyramid}} = \frac{1}{3} A_b\, h = \frac{1}{6}\sqrt{2}\,a^3.$$

$$V = V_{\text{octahedron}} - 6\,V_{\text{square pyramid}} = 8\sqrt{2}\,a^3.$$

$$Q = \frac{36\pi V^2}{S^3} = \frac{64\pi}{3\left(1 + 2\sqrt{3}\right)^3} \approx 0.753367,$$

Truncated Octahedra
tessellating space

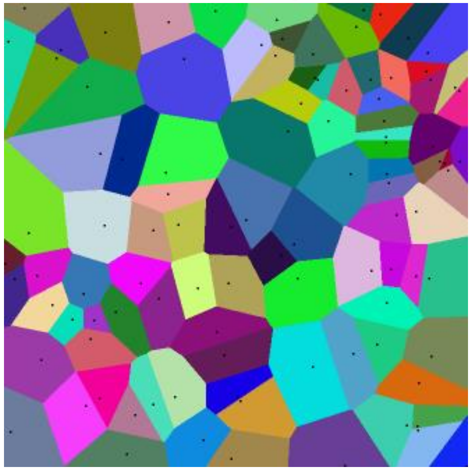
Weaire-Phelan foam



Several views of the Weaire-Phelan partition. A fundamental region of 8 different colored cells is shown. Two cells (green and blue) are dodecahedra, and the other six are 14-sided with two opposite hexagonal faces and 12 pentagonal faces. The 14-sided cells stack into three sets of orthogonal columns, and the dodecahedra fit into the interstices between the columns

Voronoi Tessellation (Diagram)

- **Given a discrete set S of points in Euclidean space**
  - Voronoi cell of point c of S:
    - ✓ is the set of all points closer to c than to any other point of S
    - ✓ A Voronoi cell is a convex polytope (polygon in 2D, polyhedron in 3D)
  - Voronoi tessellation corresponding to the s et S:
    - ✓ is the set of such polyhedra
    - ✓ tessellate the whole space
  - We assume each Voronoi cell is identical



Voronoi Diagram

Hexagonal tessellation of a floor. All cells are identical.

Analysis

- Total number of nodes for 3D coverage
  - Simply, ratio of volume to be covered to volume of one Voronoi cell
  - Minimizing no. of nodes by maximizing the volume of one cell V
  - With omnidirectional antenna: sensing range R → sphere
  - Radius of circumsphere of a Voronoi cell ≤ R
  - To achieve highest volume, radius of circumsphere = R
  - Volume of circumsphere of each Voronoi cell: $4\pi R^3/3$
  - Find space-filling polyhedron that has highest volumetric quotient; i.e., "best approximates" a sphere.
- Volumetric quotient, q: $0 \le q \le 1$
  - For any polyhedron, if the maximum distance from center to any vertex is R and the volume of the polyhedron is V, then the volumetric quotient is,

$$\frac{V}{\frac{4}{3}\pi R^3}.$$

Analysis

▪ Similarity with Kelvin's Conjecture

- Kelvin's: find space-filling polyhedron with highest *isoperimetric quotient*
  - ✓ Sphere has the highest isoperimetric quotient = 1
- Ours: find space-filling polyhedron with highest *volumetric quotient*
  - ✓ Sphere has the highest volumetric quotient = 1
- Both problems: find space-filling polyhedron "best approximates" the sphere
- Among all structures, the following claims hold:
  - ✓ For a given volume, sphere has the smallest surface area
  - ✓ For a given surface area, sphere has the largest volume

- Claim/Argument:
  - ✓ Consider two space-filling polyhedrons: P1 and P2 such that $V_{P1} = V_{P2}$
  - ✓ If $S_{P1} < S_{P2}$, then P1 is a better approximation of a sphere than P2
  - ✓ If P1 is a better approximation of a sphere than P2, then P1 has a higher volumetric quotient than P2
- Conclusion: *Solution to Kelvin's problem is essentially the solution to ours!*

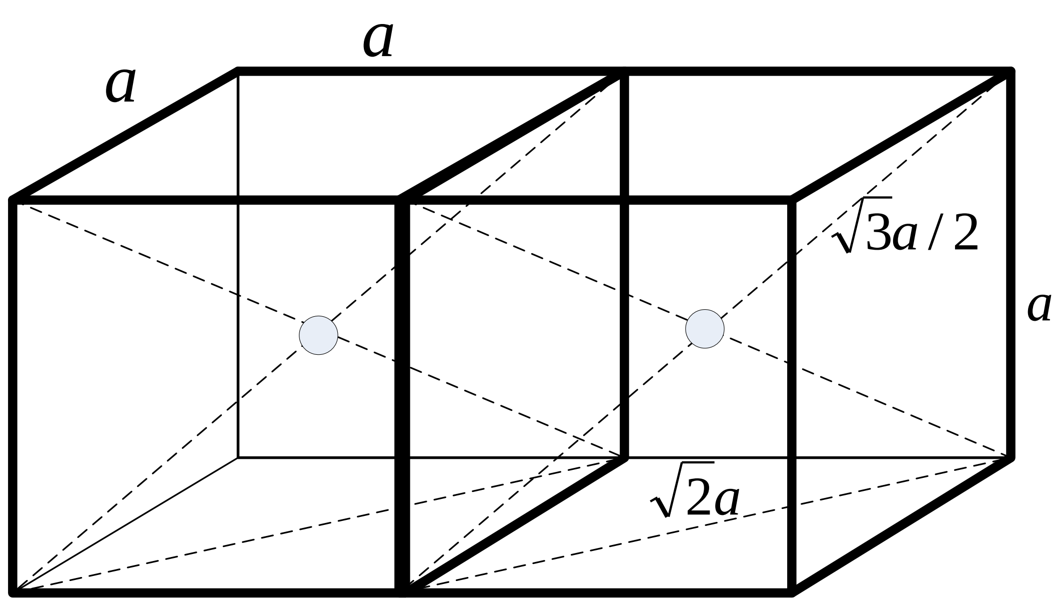
Analysis: choice of other polyhedra

- Cube
  - Simplest, only regular polyhedron tessellating 3D space
- Hexagonal prism
  - 2D optimum: hexagon, 3D extension, Used in [8]
- Rhombic dodecahedron
  - Used in [6]

- Analysis
  - Compare truncated octahedron with these polyhedra
  - Show that the truncated octahedron has a higher volumetric quotient, hence requires fewer nodes

Analysis: volumetric quotient 1

- Cube
  - Length: a
  - Radius of circumsphere = R = $\sqrt{3}a/2$
  - Volumetric quotient: $$a^3 \Big/ \frac{4}{3}\pi\left(\frac{\sqrt{3}}{2}a\right)^3 = \frac{2}{\sqrt{3}\pi} = 0.36755.$$



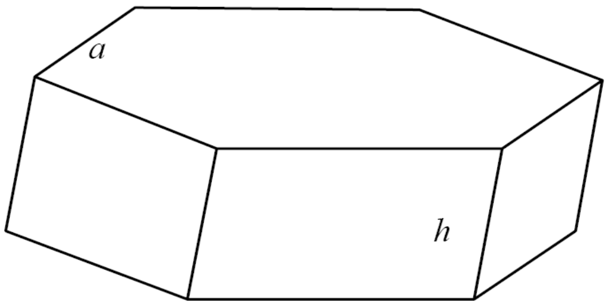
- Given R, compute a
- Sensing range: $R = \sqrt{3}a/2$
- $a = 2R/\sqrt{3} = 1.1547R$

Analysis: volumetric quotient 2

- ## Hexagonal Prism
  - Length: a, height: h
  - Volume = area of base * height
  - Radius of circumsphere = $\sqrt{a^2 + h^2/4}$
  - Volumetric quotient

$$\frac{3\sqrt{3}}{2}a^2h \Big/ \frac{4}{3}\pi\left(\sqrt{a^2+\frac{h^2}{4}}\right)^3$$

Figure 1. A Hexagonal Prism



  - Optimal h: Set first derivative of volumetric quotient to zero

$$\frac{\frac{3\sqrt{3}}{2}a^2}{\frac{4}{3}\pi\left(\sqrt{a^2+\frac{h^2}{4}}\right)^3} - \frac{3}{2}\frac{\frac{3\sqrt{3}}{2}a^2h\cdot\frac{2h}{4}}{\frac{4}{3}\pi\left(\sqrt{a^2+\frac{h^2}{4}}\right)^5} = 0$$

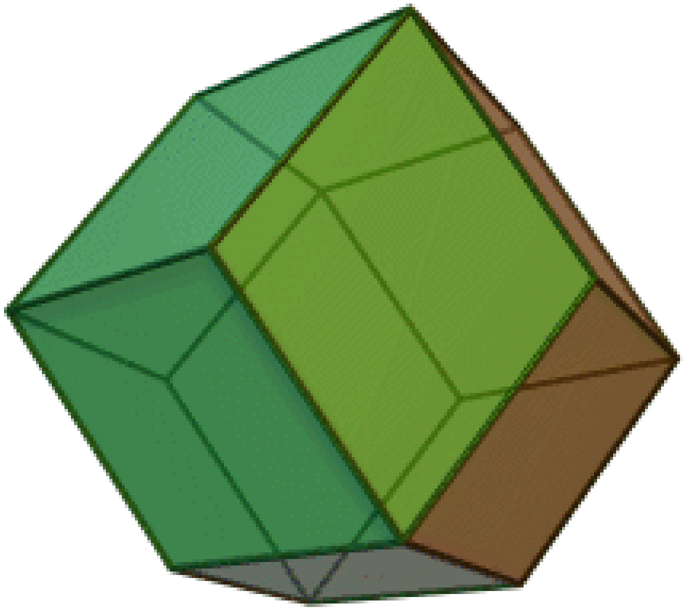
$$\Rightarrow a^2 + h^2/4 = 3h^2/4$$

$$h = a\sqrt{2}$$

  - Optim

$$\frac{3\sqrt{3}}{2}a^2a\sqrt{2} \Big/ \frac{4}{3}\pi\left(\sqrt{a^2+\frac{a^2}{2}}\right)^3 = \frac{6}{4\pi} = 0.477$$

Analysis: volumetric quotient 3



- Rhombic dodecahedron
  - 12 rhombic face
  - Length: a
  - Radius of circumsphere: a
  - Volumetric quotient

$$2a^3 \Big/ \frac{4}{3}\pi a^3 = 6/4\pi = 0.477$$

Figure 2. Construction of a rhombic dodecahedron from two identical cubes

Analysis: volumetric quotient 4

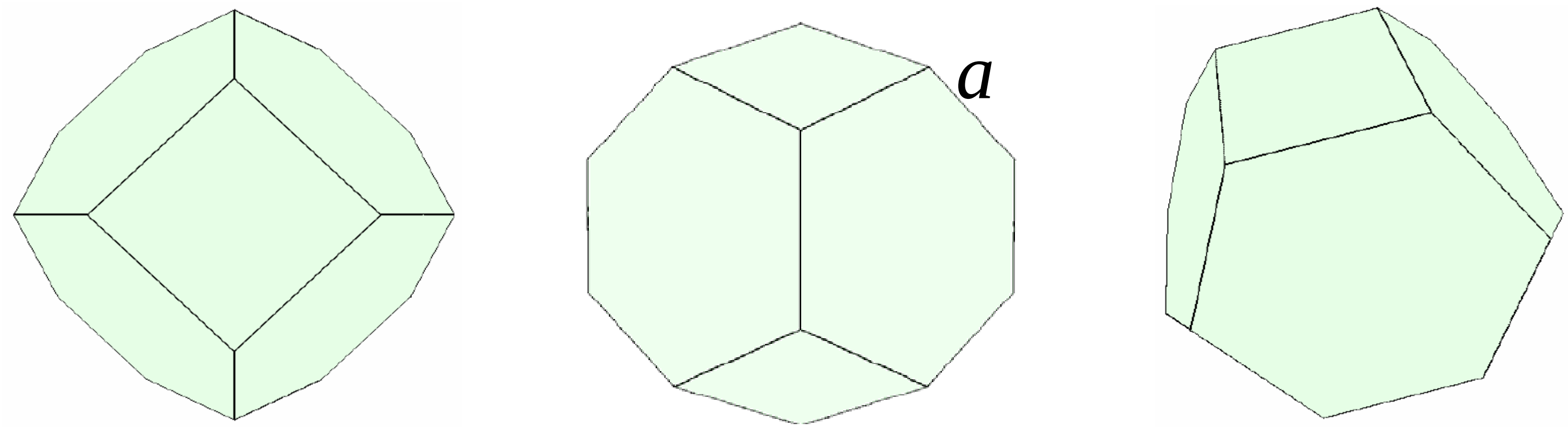


Figure 3. Truncated Octahedron.

- Truncated Octahedron
  - 14 faces, 8 hexagonal, 6 square space
  - Length: a
  - Radius of circumsphere: $\sqrt{10}a/2$
  - Volumetric quotient

$$8\sqrt{2}a^3 \Big/ \frac{4}{3}\pi\left(\frac{1}{2}\sqrt{10}a\right)^3 = 24/5\sqrt{5}\pi = 0.68329$$

Analysis: Comparison

Inverse proportion

Table I: Volumetric Quotient of Different Types of Space-filling Polyhedrons

Polyhedron	Volumetric quotient	Number of nodes needed Compared to truncated octahedron
Cube	0.36755	85.9% more
Hexagonal Prism	0.477	43.25% more
Rhombic Dodecahedron	0.477	43.25% more
Truncated Octahedron	0.68329	same

Analysis: Placement strategies

- **Node placement**
  - Where to place the nodes such that the Voronoi cells are our chosen space-filling polyhedrons?
  - Choose an arbitrary point (e.g., the center of the space to be covered): (cx, cy, cz). Place a node there.
  - Idea: Determine the locations of other nodes relative to this center node.
  - New coordinate system (u, v, w). Nodes placed at integer coordinates of this coordinate system.
  - Input to the node placement algorithm:
    - ✓ (cx, cy, cz)
    - ✓ Sensing range R
  - Output of the node placement algorithm:
    - ✓ (x, y, z) coordinates of the nodes
    - ✓ Distance between two nodes (needed to calculate transmission range. Prob. 2)

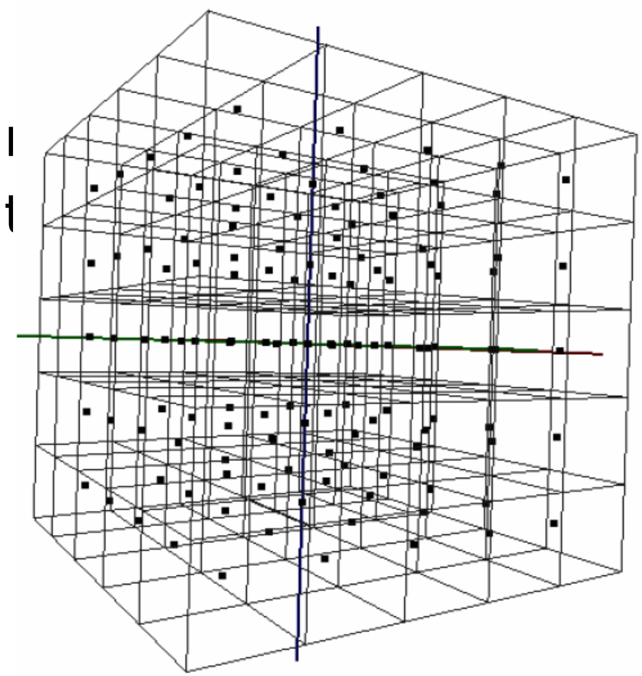
Analysis: Placement strategies 1

- Cube
  - Recall: Radius of circumsphere = R = $\sqrt{3}a/2$
  - Unit distance in each axis = a = $2R/\sqrt{3}$
  - (u, v, w) are parallel to (x, y, z)
  - A node at (u_1, v_1, w_1) in the new coordinate system should be placed i original (x, y, z) coordinate system a

$$\left(cx + u_1 \times \frac{2R}{\sqrt{3}}, cy + v_1 \times \frac{2R}{\sqrt{3}}, cz + w_1 \times \frac{2R}{\sqrt{3}} \right)$$

  - Distance between two nodes

$$d_{12}^{cb} = \frac{2}{\sqrt{3}} R \sqrt{(u_2 - u_1)^2 + (v_2 - v_1)^2 + (w_2 - w_1)^2}$$



Analysis: Placement strategies 2

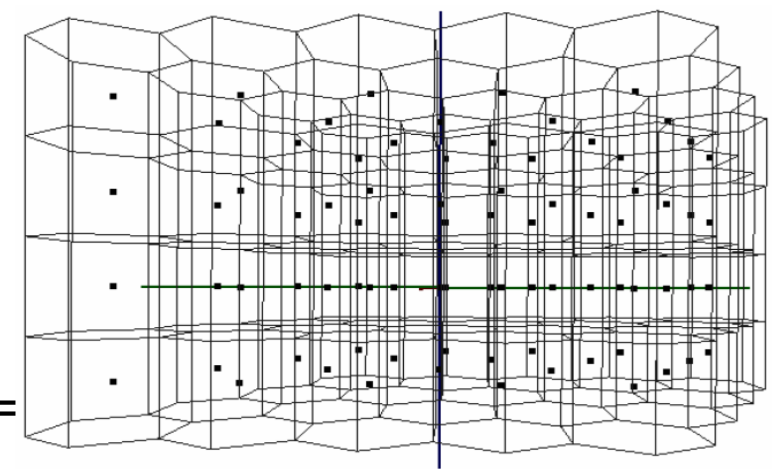
- Hexagonal Prism
  - Recall: $h = a\sqrt{2}$, $\sqrt{a^2 + h^2/4}$ $\sqrt{\frac{3}{2}}a$ =
  - Hence, a = $\sqrt{\frac{2}{3}}R$, $\frac{2R}{\sqrt{3}}$:
  - New coordinate system (u, v, w):
    - ✓ v-axis is parallel to y-axis.
    - ✓ Angle between u-axis and x-axis is 30^0
    - ✓ w-axis is parallel to z-axis
    - ✓ Unit distance along v-axis = Unit distance along $R\sqrt{2}$ =
    - ✓ Unit distance along z-axis = $\frac{2R}{\sqrt{3}}$ =

Analysis: Placement strategies 2

- **Hexagonal Prism (cont'd)**
  - A node at (u_1, v_1, w_1) in the new coordinate system should be placed in the original (x, y, z) coordinate system at

$$\left(\begin{array}{l} cx + u_1 \times R\sqrt{2}\sin 60^0, \\ cy + u_1 \times R\sqrt{2}\cos 60^0 + v_1 \times R\sqrt{2}, cz + w_1 \times \dfrac{2R}{\sqrt{3}} \end{array} \right)$$

$$= \left(cx + u_1 R\sqrt{\frac{3}{2}}, cy + (u_1 + 2v_1)\frac{R}{\sqrt{2}}, cz + \frac{2Rw_1}{\sqrt{3}} \right)$$



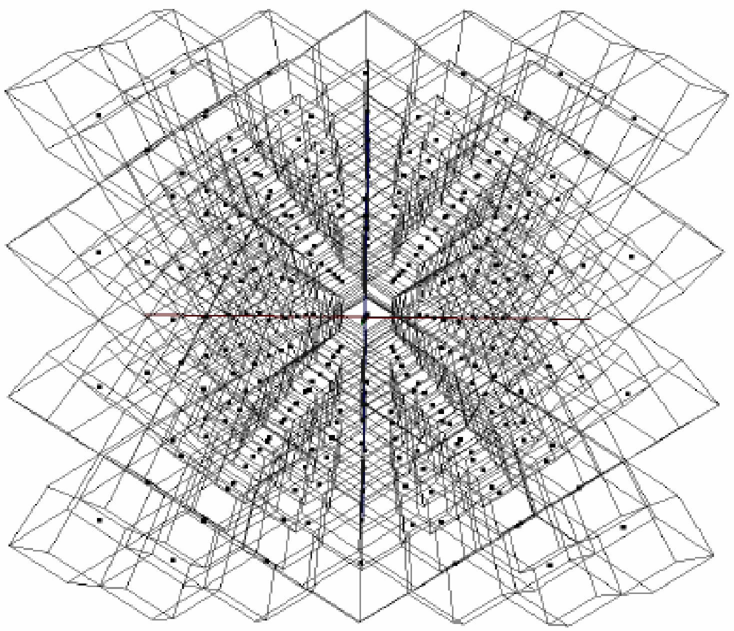
des =

$$R\sqrt{2}\sqrt{\begin{array}{l}(u_2 - u_1)^2 + (u_2 - u_1)(v_2 - v_1) + \\ (v_2 - v_1)^2 + \dfrac{2}{3}(w_2 - w_1)^2\end{array}}$$

Analysis: Placement strategies 3

- **Rhombic Dodecahedron**
  - Unit distance along each ax $R\sqrt{2}$
  - New coordinate system placed in the original coordinate system at $\left(cx+(2u_1+w_1)\frac{R}{\sqrt{2}}, cy+(2v_1+w_1)\frac{R}{\sqrt{2}}, cz+w_1R \right)$ (3)
  - Distance between two nodes

$$R\sqrt{2}\sqrt{\begin{array}{l}(u_2-u_1)^2+(v_2-v_1)^2+(w_2-w_1)^2+\\(u_2-u_1)(w_2-w_1)+(v_2-v_1)(w_2-w_1)\end{array}}$$



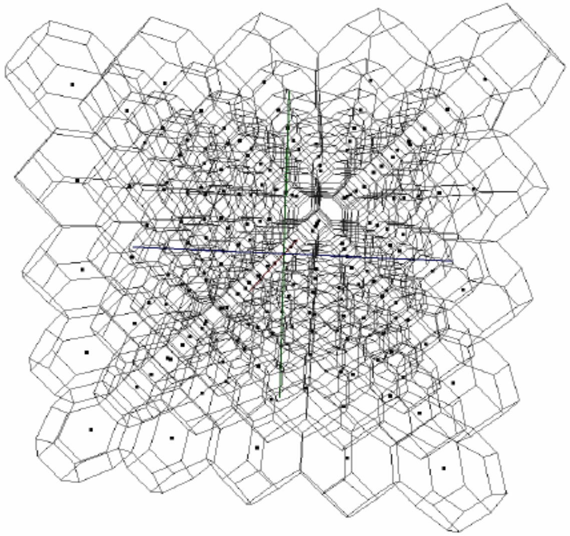
Analysis: Placement strategies 4

- **Truncated octahedron**
  - Unit distance in both u and v axes: $4R/\sqrt{5}$, w axis: $2\sqrt{3}R/\sqrt{5}$
  - New coordinate system placed in the original coordinate system at

$$\left(cx + (2u_1 + w_1)\frac{2R}{\sqrt{5}},\ cy + (2v_1 + w_1)\frac{2R}{\sqrt{5}},\ cz + w_1\frac{2R}{\sqrt{5}} \right) \quad (4)$$

  - Distance between two nodes

$$\frac{4}{\sqrt{5}}R\sqrt{\begin{aligned}&(u_2 - u_1)^2 + (v_2 - v_1)^2 + \\ &(u_2 - u_1)(w_2 - w_1) + \\ &(v_2 - v_1)(w_2 - w_1) + \frac{3}{4}(w_2 - w_1)^2\end{aligned}}$$



Analysis: Transmission vs. Sensing Range

- **Required transmission range**
  - To maintain connectivity among neighboring nodes
  - Depend on the choice of the polyhedron

Table II: Minimum Transmission Range for Different Polyhedrons

Polyhedron	Minimum Transmission Range			Max of Min Transmission Range
	u-axis	*v*-axis	*w*-axis	
Cube	$1.1547R$	$1.1547R$	$1.1547R$	$1.1547R$
Hexagonal Prism	$1.4142R$	$1.4142R$	$1.1547R$	$1.4142R$
Rhombic Dodecahedron	$1.4142R$	$1.4142R$	$1.4142R$	$1.4142R$
Truncated Octahedron	$1.7889R$	$1.7889R$	$1.5492R$	$1.7889R$

Conclusion

- Performance comparison
  - Truncated octahedron: higher volumetric quotient (0.683) than others(0.477, 0.367)
  - Required much fewer nodes (others more than 43%)
- Maintain full connectivity
  - Optimal placement strategy for each polyhedron
  - Truncated octahedron: requires the transmission range to be at least 1.7889 times the sensing range
- Further applications
  - Fixed: initial node deployment
  - Mobile: dynamically place to desired location
  - Node ID: u,v,w coordination ⇨ location-based routing protocol