Νευρο-Ασαφής Υπολογιστική
Neuro-Fuzzy Computing

Διδάσκων –

Δημήτριος Κατσαρός

@ Τμ. ΗΜΜΥ

Πανεπιστήμιο Θεσσαλίας

Διάλεξη 26η

Fuzzy Subset Theory

Basic set-theoretic operations

The concept of a fuzzy subset

- Let **E** be a set denumerable or not, and let x be an element of **E**. A *fuzzy subset* \utilde{A} of **E** is a set of ordered pairs:

$$\{(x|\mu_{\utilde{A}}(x))\}, \forall x \in E$$

  where $\mu_{\utilde{A}}(x)$ is a *membership characteristic function* that takes its values in a totally ordered set **M**, and which indicates the *degree* or *level of membership*. **M** will be called a *membership set*

- If **M**={0, 1}, the "fuzzy subset" \utilde{A} will be understood as an "ordinary subset"
- Cardinality of a fuzzy subset: $|\utilde{A}| = \sum_{x \in E} \mu_{\utilde{A}}(x)$
- Examples:
  I. The fuzzy subset of numbers x approximately equal to a given real number n
  II. The fuzzy subset of integers near to 0
  III. The fuzzy subset of integers very near to 0

The concept of a fuzzy subset

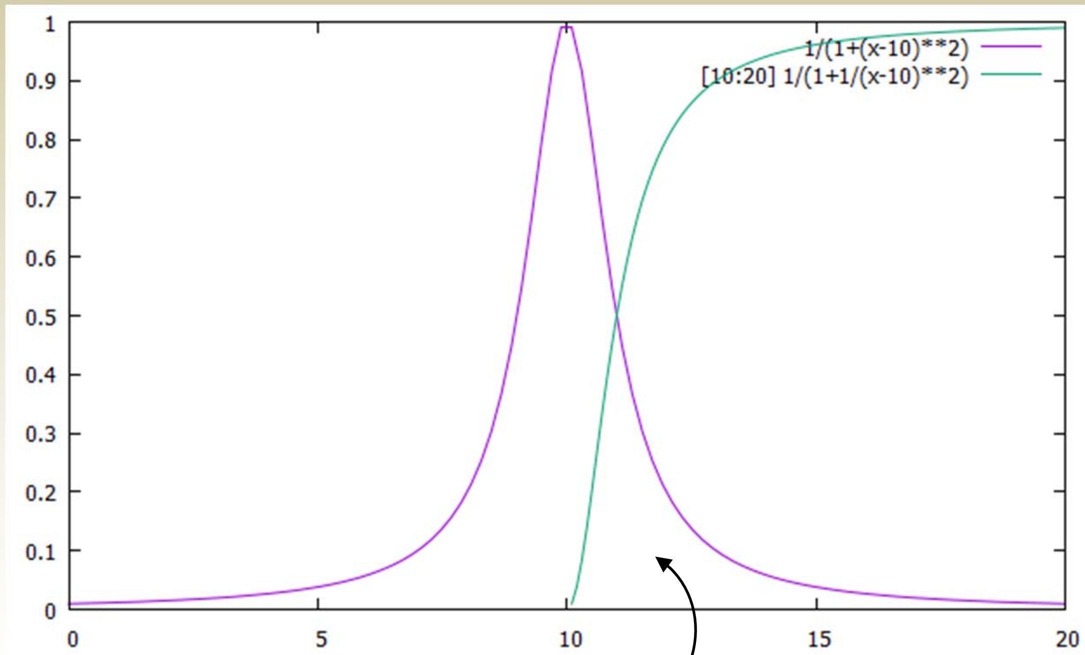
- Examples:

IV. The fuzzy subset $\underset{\sim}{A}$ of real numbers close to 10

V. The fuzzy subset $\underset{\sim}{B}$ of real numbers significantly larger than 10

$$\mu_{\underset{\sim}{A}}(x) = \frac{1}{1 + (x - 10)^2}$$

$$\mu_{\underset{\sim}{B}}(x) = \begin{cases} 0, & \text{if } \; x \leq 10 \\ \frac{1}{1 + \frac{1}{(x-10)^2}} & x > 10 \end{cases}$$



$\underset{\sim}{A} \cap \underset{\sim}{B}$

Real numbers who are close to 10 *fuzzy-and* significantly larger than 10

Simple operations on fuzzy subsets

- **Inclusion**: We say that \utilde{A} is included in \utilde{B} if

$$\forall x \in E : \mu_{\utilde{A}}(x) \leq \mu_{\utilde{B}}(x) \quad \text{denoted as} \quad \utilde{A} \subset \utilde{B} \text{ or } \utilde{A} \utilde{\subseteq} \utilde{B}$$

$$\text{We can write: } \quad E \utilde{\subseteq} E$$

- **Strict inclusion**: If for at least one x, it holds that: $\mu_{\text{\utilde{A}}}(x) < \mu_{\text{\utilde{B}}}(x)$

$$\text{denoted as} \quad \utilde{A} \subset\subset \utilde{B}$$

- **Equality**:

$$\forall x \in E : \mu_{\utilde{A}}(x) = \mu_{\utilde{B}}(x) \quad \text{denoted as} \quad \utilde{A} = \utilde{B}$$

- **Complementation** (actually, it is *pseudo-complementation*):

$$\forall x \in E : \mu_{\utilde{B}}(x) = 1 - \mu_{\utilde{A}}(x) \quad \text{denoted as} \quad \utilde{B} = \overline{\utilde{A}}$$

$$\text{It holds that: } \quad \overline{(\overline{\utilde{A}})} = A$$

Simple operations on fuzzy subsets

- **<u>Intersection</u>**: This is the *fuzzy and*

$$\forall x \in E : \mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(x) = MIN\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\} \quad \text{denoted as} \quad \underset{\sim}{A} \cap \underset{\sim}{B}$$

- **<u>Union</u>**: This is the *fuzzy or/and*

$$\forall x \in E : \mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(x) = MAX\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\} \quad \text{denoted as} \quad \underset{\sim}{A} \cup \underset{\sim}{B}$$

- **<u>Disjunctive sum</u>**: This is the *fuzzy disjunctive or*

$$\underset{\sim}{A} \oplus \underset{\sim}{B} = (\underset{\sim}{A} \cap \overline{\underset{\sim}{B}}) \cup (\overline{\underset{\sim}{A}} \cap \underset{\sim}{B})$$

- **<u>Difference</u>**:

$$\underset{\sim}{A} - \underset{\sim}{B} = \underset{\sim}{A} \cap \overline{\underset{\sim}{B}}$$

Simple operations on fuzzy subsets

- **Generalized Hamming distance**:

$$d(\underset{\sim}{A}, \underset{\sim}{B}) = \sum_{i=1}^{n} \left| \mu_{\underset{\sim}{A}}(x_i) - \mu_{\underset{\sim}{B}}(x_i) \right|$$

- **Euclidean distance or Quadratic distance**:

$$e(\underset{\sim}{A}, \underset{\sim}{B}) = \sqrt{\sum_{i=1}^{n} \left(\mu_{\underset{\sim}{A}}(x_i) - \mu_{\underset{\sim}{B}}(x_i) \right)^2}$$

- **Generalized relative Hamming distance**:

$$\delta(\underset{\sim}{A}, \underset{\sim}{B}) = \frac{d(\underset{\sim}{A}, \underset{\sim}{B})}{n}$$

- **Relative Euclidean distance**:

$$\epsilon(\underset{\sim}{A}, \underset{\sim}{B}) = \frac{e(\underset{\sim}{A}, \underset{\sim}{B})}{\sqrt{n}}$$

Simple operations on fuzzy subsets

- **<u>The ordinary subset nearest to a fuzzy subset</u>**:

$$\begin{aligned} \mu_{\underline{\underset{\sim}{A}}}(x_i) &= 0 && \text{if } \mu_{\underset{\sim}{A}}(x_i) < 0.5 \\ &= 1 && \text{if } \mu_{\underset{\sim}{A}}(x_i) > 0.5 \\ &= 0 \text{ or } 1 && \text{if } \mu_{\underset{\sim}{A}}(x_i) = 0.5 \end{aligned}$$

- **<u>Index of fuzziness</u>**:
  - Linear index of fuzziness

$$\nu(\underset{\sim}{A}) = \frac{2}{n} d(\underset{\sim}{A}, \underline{\underset{\sim}{A}})$$

  - Quadratic index of fuzziness:

$$\eta(\underset{\sim}{A}) = \frac{2}{\sqrt{n}} e(\underset{\sim}{A}, \underline{\underset{\sim}{A}})$$

Simple operations on fuzzy subsets

- **Properties concerning the nearest ordinary subset**:

$$\underline{\underset{\sim}{A} \cap \underset{\sim}{B}} = \underline{\underset{\sim}{A}} \cap \underline{\underset{\sim}{B}}$$

$$\underline{\underset{\sim}{A} \cup \underset{\sim}{B}} = \underline{\underset{\sim}{A}} \cup \underline{\underset{\sim}{B}}$$

$$\forall x_i \in E : |\mu_{\underset{\sim}{A}}(x_i) - \mu_{\underline{\underset{\sim}{A}}}(x_i)| = \mu_{\underset{\sim}{A} \cap \overline{\underset{\sim}{A}}}(x_i)$$

- One sometimes calls the fuzzy subset whose membership function is $2\mu_{\underset{\sim}{A} \cap \overline{\underset{\sim}{A}}}(x)$ the *vectorial indicator of fuzziness*

Simple operations on fuzzy subsets

- **Evaluation of *fuzziness through entropy***
- Recall the entropy of a system comprised by N states:

$$\mathcal{H}(p_1, p_2, \ldots, p_N) = -\sum_{i=1}^{N} p_i \times ln(p_i)$$

- minimum value= 0, maximum value= ln(N)
- Thus, the above equation in [0,1] becomes a measure of fuzziness:

$$\mathcal{H}(p_1, p_2, \ldots, p_N) = -\frac{1}{ln(N)}\sum_{i=1}^{N} p_i \times ln(p_i)$$

- Explanation through an example:

$$\mu_{\underset{\sim}{A}}(x_1) = 0.7, \quad \mu_{\underset{\sim}{A}}(x_2) = 0.9, \quad \mu_{\underset{\sim}{A}}(x_3) = 0.0,$$

$$\mu_{\underset{\sim}{A}}(x_4) = 0.6, \quad \mu_{\underset{\sim}{A}}(x_5) = 0.5, \quad \mu_{\underset{\sim}{A}}(x_6) = 1,$$

Simple operations on fuzzy subsets

- Putting:

$$\pi_{\underset{\sim}{A}}(x_i) = \frac{\mu_{\underset{\sim}{A}}(x_i)}{\sum_{i=1}^{6} \mu_{\underset{\sim}{A}}(x_i)}$$

- We get:

$$\pi_{\underset{\sim}{A}}(x_1) = \frac{7}{37}, \quad \pi_{\underset{\sim}{A}}(x_2) = \frac{9}{37}, \quad \pi_{\underset{\sim}{A}}(x_3) = 0.0,$$

$$\pi_{\underset{\sim}{A}}(x_4) = \frac{6}{37}, \quad \pi_{\underset{\sim}{A}}(x_5) = \frac{5}{37}, \quad \pi_{\underset{\sim}{A}}(x_6) = \frac{10}{37}$$

Therefore:

$$\mathcal{H}(\pi_1, \pi_2, \ldots, \pi_6) = -\frac{1}{ln(6)} \sum_{i=1}^{6} \pi_{\underset{\sim}{A}}(x_i) \times ln(\pi_{\underset{\sim}{A}}(x_i)) = \cdots = 0.89$$

Entropy may be used in the theory of fuzzy subsets, but it is not a good indicator

Simple operations on fuzzy subsets

- **Ordinary subset of level α**:
  - For α ϵ [0,1]

$$A_\alpha = \{x | \mu_{\utilde{A}}(x) \geq \alpha\}$$

  - Important property:

$$\alpha_2 \geq \alpha_1 \Rightarrow A_{a_2} \subset A_{a_1}$$

- **Decomposition theorem**: Any fuzzy subset \utilde{A} can be decomposed as products of ordinary subsets by the coefficients α_i

$$\utilde{A} = \max_{\alpha_i} [\alpha_1 \times A_{\alpha_1}, \alpha_2 \times A_{\alpha_2}, \ldots, \alpha_n \times A_{\alpha_n}],$$

$$0 < \alpha_i \leq 1, \quad i = 1, 2, \ldots, n$$

Decomposition theorem proof

- **Proof**: The proof is immediate:

$$\mu_{A_{\alpha_i}}(x) = 1, \text{ if } \mu_{\underset{\sim}{A}}(x) \geq \alpha_i$$

$$\mu_{A_{\alpha_i}}(x) = 0, \text{ if } \mu_{\underset{\sim}{A}}(x) < \alpha_i$$

- So, the membership function of \utilde{A} may be written:

$$\underset{\max_{\alpha_i}[\alpha_i A_{\alpha_i}]}{\mu(x)} = \max_{\alpha_i}[\alpha_i A_{\alpha_i}]$$

$$= \max_{\alpha_i \leq \mu_{\underset{\sim}{A}}(x)} [\alpha_i]$$

$$= \mu_{\underset{\sim}{A}}(x)$$

Decomposition theorem example

$$
\begin{array}{|c|c|c|c|c|}
\multicolumn{1}{c}{x_1} & \multicolumn{1}{c}{x_2} & \multicolumn{1}{c}{x_3} & \multicolumn{1}{c}{x_4} & \multicolumn{1}{c}{x_5} \\
\hline
0{,}2 & 0 & 0{,}5 & 1 & 0{,}7 \\
\hline
\end{array}
= \mathrm{MAX}\left((0{,}2) \, .
\begin{array}{|c|c|c|c|c|}
\multicolumn{1}{c}{x_1} & \multicolumn{1}{c}{x_2} & \multicolumn{1}{c}{x_3} & \multicolumn{1}{c}{x_4} & \multicolumn{1}{c}{x_5} \\
\hline
1 & 0 & 1 & 1 & 1 \\
\hline
\end{array}
, \right.
$$

$$
(0{,}5) \, .
\begin{array}{|c|c|c|c|c|}
\multicolumn{1}{c}{x_1} & \multicolumn{1}{c}{x_2} & \multicolumn{1}{c}{x_3} & \multicolumn{1}{c}{x_4} & \multicolumn{1}{c}{x_5} \\
\hline
0 & 0 & 1 & 1 & 1 \\
\hline
\end{array}
, \quad (0{,}7) \, .
\begin{array}{|c|c|c|c|c|}
\multicolumn{1}{c}{x_1} & \multicolumn{1}{c}{x_2} & \multicolumn{1}{c}{x_3} & \multicolumn{1}{c}{x_4} & \multicolumn{1}{c}{x_5} \\
\hline
0 & 0 & 0 & 1 & 1 \\
\hline
\end{array}
,
$$

$$
\left. (1) \, .
\begin{array}{|c|c|c|c|c|}
\multicolumn{1}{c}{x_1} & \multicolumn{1}{c}{x_2} & \multicolumn{1}{c}{x_3} & \multicolumn{1}{c}{x_4} & \multicolumn{1}{c}{x_5} \\
\hline
0 & 0 & 0 & 1 & 0 \\
\hline
\end{array}
\right) .
$$

Set of fuzzy subsets for E and M finite

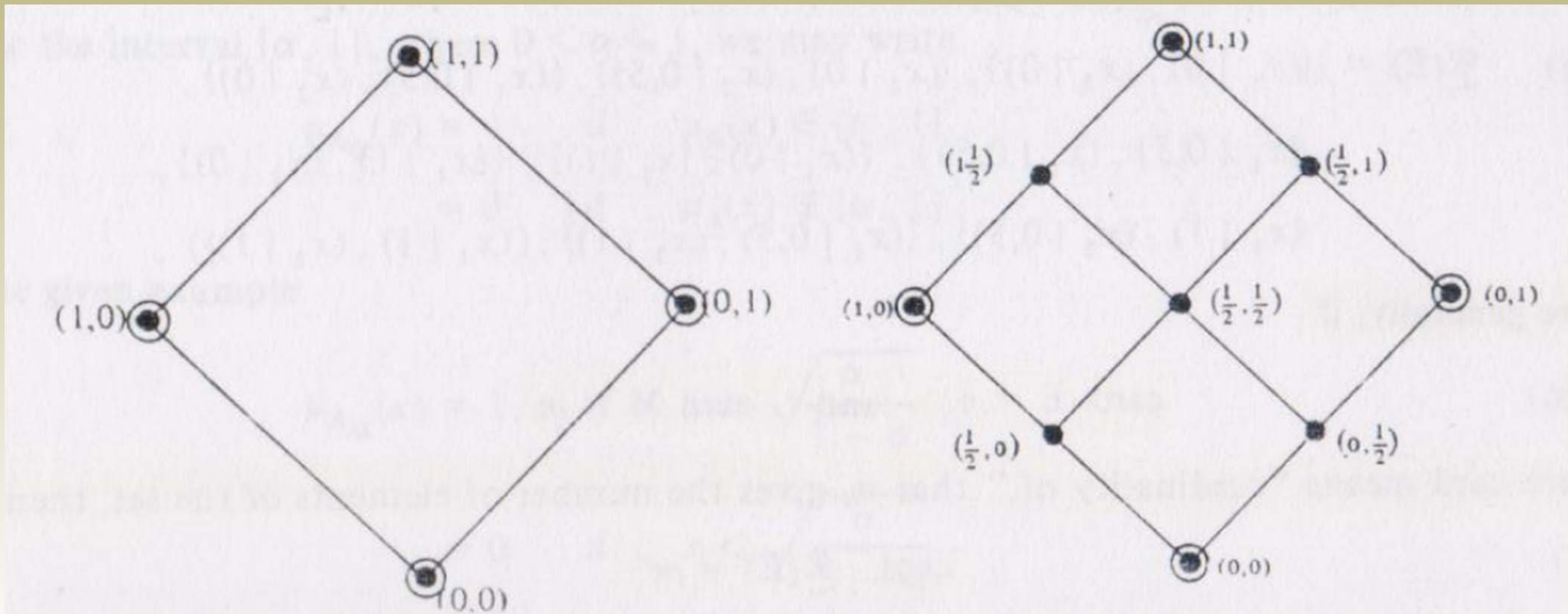
- **<u>The powerset for a fuzzy subset</u>**

  - If cardinality[**E**]= *n* and cardinality[**M**]= *m*, then:

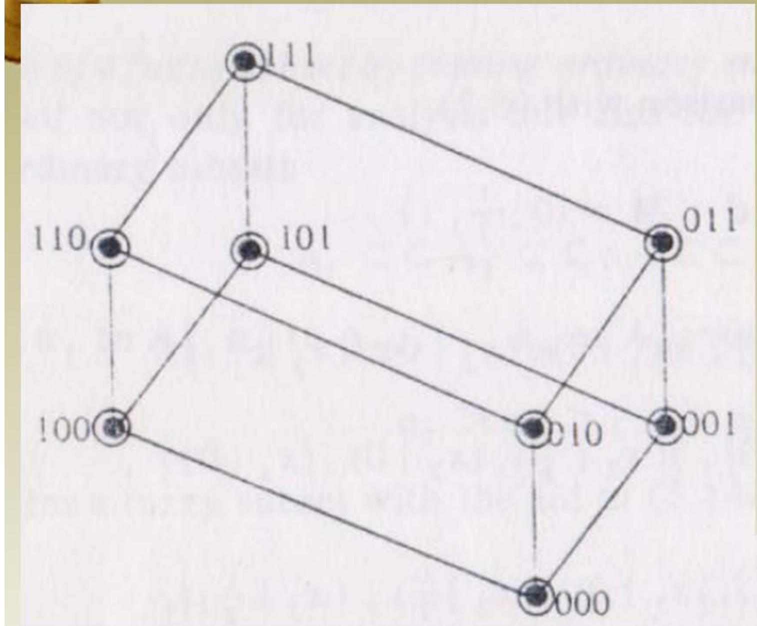
$$cardinality[\underset{\sim}{\mathcal{P}}(E)] = m^n$$

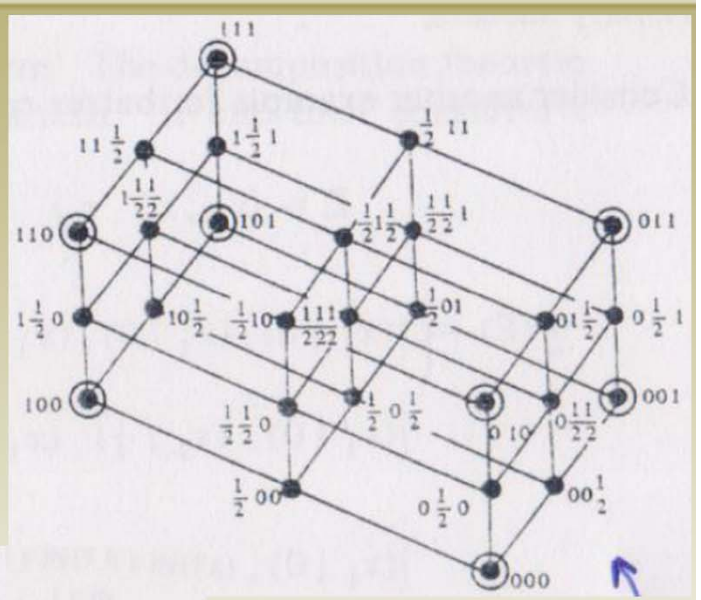
  - It is well known that the structure of a power set \mathcal{\utilde{P}}(E) of a set is a distributive and complementary lattice, that is, a *boolean lattice*. The set of fuzzy subsets \mathcal{\utilde{P}}(E), however, has the structure of a *vectorial lattice* that is distributive but not complimentary

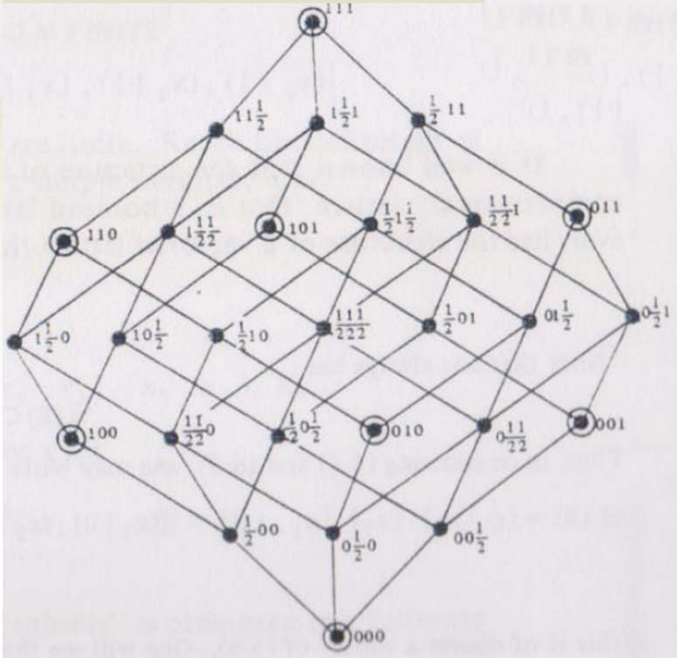
Set of fuzzy subsets for E and M finite



Set of fuzzy subsets for E and M finite







Properties of the powerset of ordinary set

$A \cap B = B \cap A$	(1)	commutativity properties
$A \cup B = B \cup A$	(2)	
$(A \cap B) \cap C = A \cap (B \cap C)$	(3)	associativity properties
$(A \cup B) \cup C = A \cup (B \cup C)$	(4)	
$A \cap A = A$	(5)	idempotence
$A \cup A = A$	(6)	
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	(7)	distributivity of intersection with respect to union, and of union with respect to intersection
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(8)	
$A \cap \overline{A} = \emptyset$ **Law of contradiction**	(9)	
$A \cup \overline{A} = E$ **Law of excluded middle**	(10)	
$A \cap \emptyset = \emptyset$	(11)	absorption by ∅
$A \cup \emptyset = A$	(12)	identity
$A \cap E = A$	(13)	
$A \cup E = E$	(14)	absorption by **E**
$\overline{(\overline{A})} = A$	(15)	involution
$\overline{A \cap B} = \overline{A} \cup \overline{B}$	(16)	De Morgan's theorems
$\overline{A \cup B} = \overline{A} \cap \overline{B}$	(17)	

Properties of the set of fuzzy subsets

$$\underset{\sim}{A} \cap \underset{\sim}{B} = \underset{\sim}{B} \cap \underset{\sim}{A} \tag{18}$$

$$\underset{\sim}{A} \cup \underset{\sim}{B} = \underset{\sim}{B} \cup \underset{\sim}{A} \tag{19}$$

commutativity properties

$$(\underset{\sim}{A} \cap \underset{\sim}{B}) \cap \underset{\sim}{C} = \underset{\sim}{A} \cap (\underset{\sim}{B} \cap \underset{\sim}{C}) \tag{20}$$

$$(\underset{\sim}{A} \cup \underset{\sim}{B}) \cup \underset{\sim}{C} = \underset{\sim}{A} \cup (\underset{\sim}{B} \cup \underset{\sim}{C}) \tag{21}$$

associativity properties

$$\underset{\sim}{A} \cap \underset{\sim}{A} = \underset{\sim}{A} \tag{22}$$

$$\underset{\sim}{A} \cup \underset{\sim}{A} = \underset{\sim}{A} \tag{23}$$

idempotence

$$\underset{\sim}{A} \cap (\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cup (\underset{\sim}{A} \cap \underset{\sim}{C}) \tag{24}$$

$$\underset{\sim}{A} \cup (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cap (\underset{\sim}{A} \cup \underset{\sim}{C}) \tag{25}$$

distributivity of intersection with respect to union, and of union with respect to intersection

$$\underset{\sim}{A} \cap \emptyset = \emptyset \tag{26}$$

$$\underset{\sim}{A} \cup \emptyset = \underset{\sim}{A} \tag{27}$$

where $\varnothing$ is the ordinary set, such that $\mu_{\varnothing}(x_i)=0, \forall x_i$

$$\underset{\sim}{A} \cap E = \underset{\sim}{A} \tag{28}$$

$$\underset{\sim}{A} \cup E = E \tag{29}$$

where **E** is the ordinary set, such that $\mu_{\mathbf{E}}(x_i)=1, \forall x_i$

$$\overline{(\overline{\underset{\sim}{A}})} = \underset{\sim}{A} \tag{30}$$

involution

$$\overline{\underset{\sim}{A} \cap \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}} \tag{31}$$

$$\overline{\underset{\sim}{A} \cup \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{B}} \tag{32}$$

De Morgan's theorems

Properties of the set of fuzzy subsets

- We see that all properties (1)-(17) are satisfied except from (9) and (10)
- One may define a unique complement, but the properties (9) and (10) hold only for ordinary subsets

- Thus, we stress: All properties of an ordinary power set are found again in a power set of fuzzy subsets, except (9) and (10). Thus, we no longer have an *algebra* in the sense of the theory of ordinary sets; the structure is that of a *vector lattice*

Algebraic product and sum of two fuzzy subsets

- **Algebraic product**: **E** be an ordinary set and **M**=[0, 1]

$$\forall x \in E : \mu_{\underset{\sim}{A}.\underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \times \mu_{\underset{\sim}{B}}(x) \quad \text{denoted as} \quad \underset{\sim}{A}.\underset{\sim}{B}$$

- **Algebraic sum**:

$$\forall x \in E : \mu_{\underset{\sim}{A}\hat{+}\underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(x) - \mu_{\underset{\sim}{A}}(x) \times \mu_{\underset{\sim}{B}}(x) \quad \text{denoted as} \quad \underset{\sim}{A}\hat{+}\underset{\sim}{B}$$

- One important remark:
  - If **M**={0, 1}, i.e., we are in the case of ordinary subsets, then

$$A \cap B = A.B$$

$$A \cup B = A \hat{+} B$$

Algebraic product and sum of two fuzzy subsets

$$\underset{\sim}{A}.\underset{\sim}{B} = \underset{\sim}{B}.\underset{\sim}{A} \tag{33}$$

$$\underset{\sim}{A}\hat{+}\underset{\sim}{B} = \underset{\sim}{B}\hat{+}\underset{\sim}{A} \tag{34}$$

(33)–(34): commutativity properties

$$(\underset{\sim}{A}.\underset{\sim}{B}).\underset{\sim}{C} = \underset{\sim}{A}.(\underset{\sim}{B}.\underset{\sim}{C}) \tag{35}$$

$$(\underset{\sim}{A}\hat{+}\underset{\sim}{B})\hat{+}\underset{\sim}{C} = \underset{\sim}{A}\hat{+}(\underset{\sim}{B}\hat{+}\underset{\sim}{C}) \tag{36}$$

(35)–(36): associativity properties

$$\underset{\sim}{A}.\emptyset = \emptyset \tag{37}$$

$$\underset{\sim}{A}\hat{+}\emptyset = \underset{\sim}{A} \tag{38}$$

$$\underset{\sim}{A}.E = \underset{\sim}{A} \tag{39}$$

$$\underset{\sim}{A}\hat{+}E = E \tag{40}$$

$$\overline{(\overline{\underset{\sim}{A}})} = \underset{\sim}{A} \tag{41}$$

(41): involution

$$\overline{\underset{\sim}{A}.\underset{\sim}{B}} = \overline{\underset{\sim}{A}}\hat{+}\overline{\underset{\sim}{B}} \tag{42}$$

$$\overline{\underset{\sim}{A}\hat{+}\underset{\sim}{B}} = \overline{\underset{\sim}{A}}.\overline{\underset{\sim}{B}} \tag{43}$$

(42)–(43): De Morgan’s theorems

- Only the above properties are verified. Idempotence [(5) and (6)], distributivity [(7) and (8)], and of course (9) and (10) are not satisfied

Algebraic product and sum of two fuzzy subsets

- Note that $\cup$ is not distributive with respect to . or $\hat{+}$, and likewise $\cap$, but on the other hand one has:

$$\underset{\sim}{A}.(\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A}.\underset{\sim}{B}) \cap (\underset{\sim}{A}.\underset{\sim}{C}) \tag{44}$$

$$\underset{\sim}{A}.(\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A}.\underset{\sim}{B}) \cup (\underset{\sim}{A}.\underset{\sim}{C}) \tag{45}$$

$$\underset{\sim}{A}\hat{+}(\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A}\hat{+}\underset{\sim}{B}) \cap (\underset{\sim}{A}\hat{+}\underset{\sim}{C}) \tag{46}$$

$$\underset{\sim}{A}\hat{+}(\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A}\hat{+}\underset{\sim}{B}) \cup (\underset{\sim}{A}\hat{+}\underset{\sim}{C}) \tag{47}$$

Algebraic product and sum of two fuzzy subsets

- Let us prove (42):
  - Suppose that $\mu_A(x)=a$ and $\mu_B(x)=b$
  - The left part gives: 1-ab
  - The right part gives: (1-a)+(1-b)-(1-a)(1-b)= 1-a+1-b-1-ab+a+b= 1-ab
  - Thus, the two parts are alike
- Let us disprove that distributivity holds, i.e., that

$$\underset{\sim}{A}.(\underset{\sim}{B}\hat{+}\underset{\sim}{C}) \neq (\underset{\sim}{A}.\underset{\sim}{B})\hat{+}(\underset{\sim}{A}.\underset{\sim}{C})$$

  - The left part gives: a (b+c-bc)= ab +ac -abc
  - The right part gives: ab + ac –abac= ab +ac –a^2bc

Fuzzy relation

- Example 1
  - $E_1 = \{x_1, x_2, x_3\}$
  - $E_2 = \{y_1, y_2, y_3, y_4, y_5\}$
  - M= [0, 1]

	y_1	y_2	y_3	y_4	y_5
x_1	0	0	0,1	0,3	1
x_2	0	0,8	0	0	1
x_3	0,4	0,4	0,5	0	0,2

- Example 2
  - $E_1 = E_2 = R$
  - Η σχέση: y << x is a fuzzy relation

$$\mu_{R^2}(x,y) = \begin{cases} 0, & \text{if } \; y \geq x \\ \frac{1}{1+\frac{1}{(x-y)^2}} & y < x \end{cases}$$

Projection of a fuzzy relation

- *First projection* of \utilde{R}
- *Second projection* of \utilde{R}
- The second projection of the first projection (or vice versa) will be called the *global projection*

$$\mu^{(1)}_{\underset{\sim}{R}}(x) = \bigvee_{y} \mu_{\underset{\sim}{R}}(x, y)$$

$$\mu^{(2)}_{\underset{\sim}{R}}(y) = \bigvee_{x} \mu_{\underset{\sim}{R}}(x, y)$$

$$h(\underset{\sim}{R}) = \bigvee_{x} \bigvee_{y} \mu_{\underset{\sim}{R}}(x, y)$$

$$= \bigvee_{y} \bigvee_{x} \mu_{\underset{\sim}{R}}(x, y)$$

$\underset{\sim}{R}$	y_1	y_2	y_3	y_4	1st proj.
x_1	0,1	0,2	1	0,3	1
x_2	0,6	0,8	0	0,1	0,8
x_3	0	1	0,3	0,6	1
x_4	0,8	0,1	1	0	1
x_5	0,9	0,7	0	0,5	0,9
x_6	0,9	0	0,3	0,7	0,9
2nd proj.	0,9	1	1	0,7	1 (global projection)

If h(\utilde{R}) = 1, the relation is said to be *normal*.
If h(\utilde{R}) < 1, the relation is called *subnormal*.

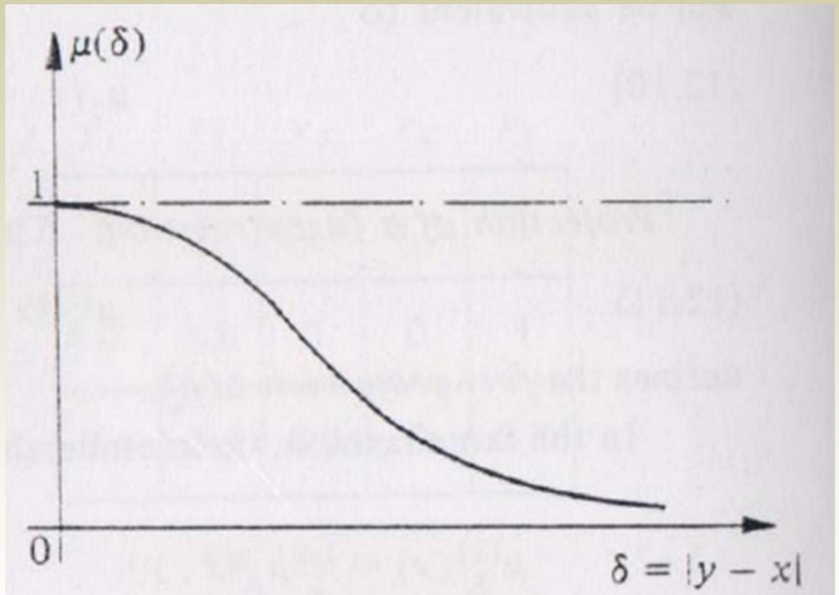
Projection of a fuzzy relation: Example 2

- x and y are very near to one another:

$$\mu_{\underset{\sim}{R}}(x, y) = e^{-k(y-x)^2}, \quad k > 1$$

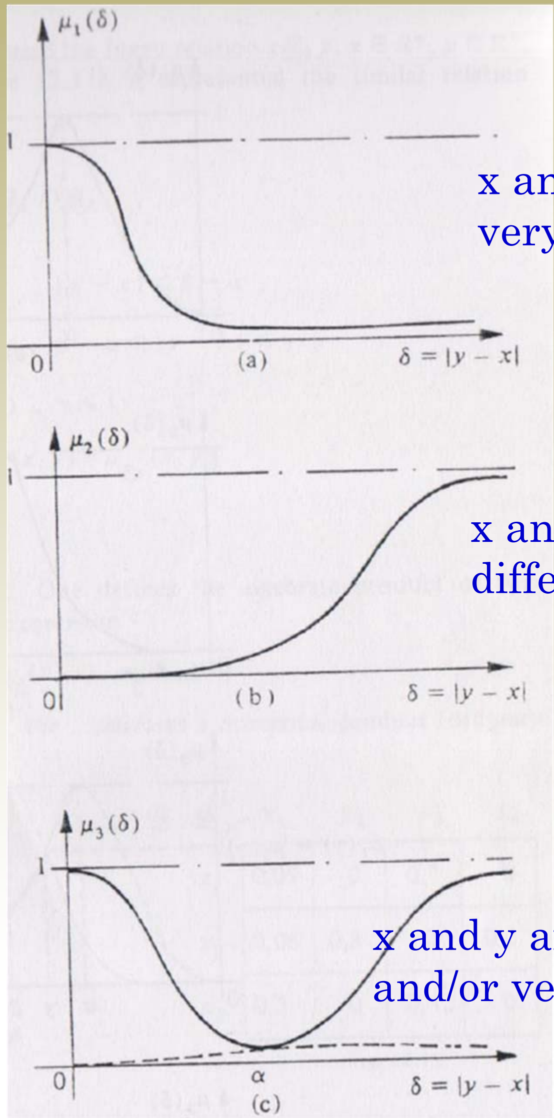
- For a fixed value x_0:

$$\begin{aligned} \mu_{\underset{\sim}{R}}^{(1)}(x_0) &= \underset{y}{V} \mu_{\underset{\sim}{R}}(x_0, y) \\ &= \underset{y}{V} e^{-k(y-x_0)^2} \\ &= e^{-k(y-x_0)^2} \quad \text{for } y = x_0 \\ &= 1 \end{aligned}$$



Union of two fuzzy relations

$$\mu_{\underset{\sim}{R}\cup\underset{\sim}{Q}}(x,y) = \mu_{\underset{\sim}{R}}(x,y) \vee \mu_{\underset{\sim}{Q}}(x,y)$$
$$= \max[\mu_{\underset{\sim}{R}}(x,y), \mu_{\underset{\sim}{Q}}(x,y)]$$



x and y are very near

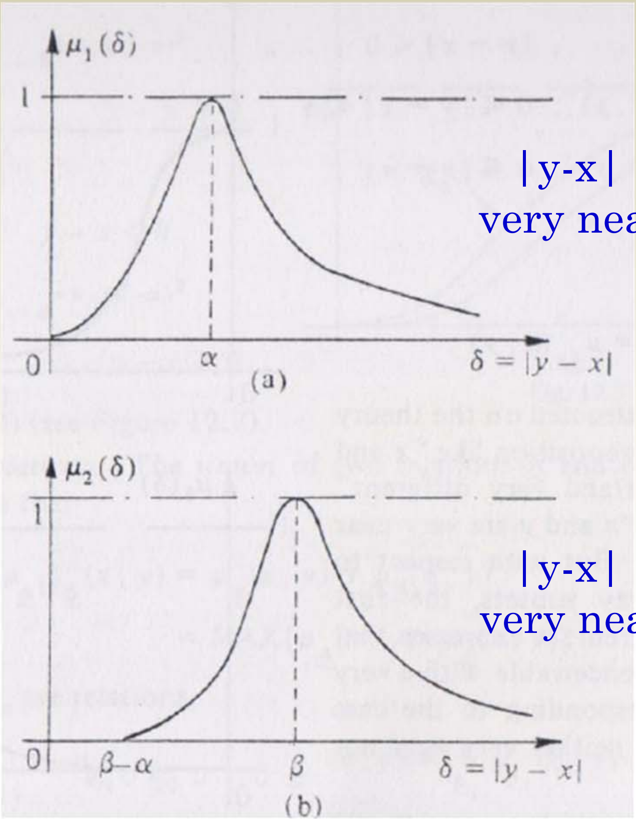
x and y are very different

x and y are very near and/or very different

Intersection of two fuzzy relations

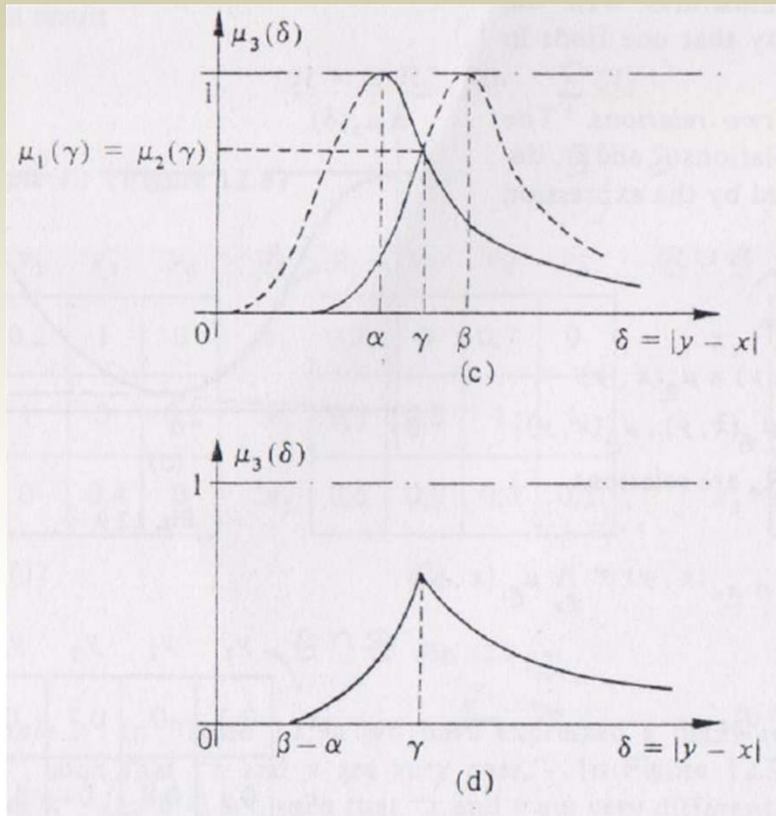
$$\mu_{\underset{\sim}{R} \cap \underset{\sim}{Q}}(x, y) = \mu_{\underset{\sim}{R}}(x, y) \wedge \mu_{\underset{\sim}{Q}}(x, y)$$

$$= \min[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{Q}}(x, y)]$$



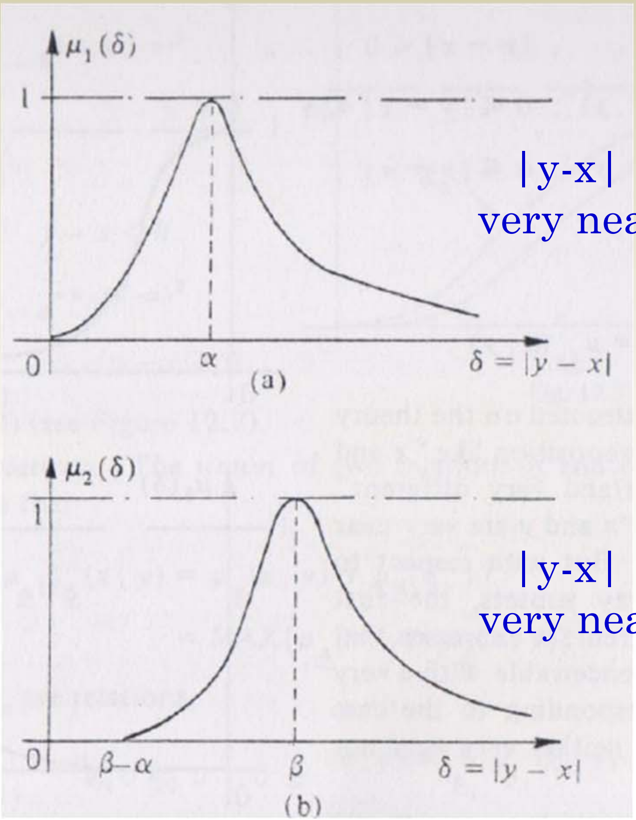
|y-x| is very near α

|y-x| is very near β



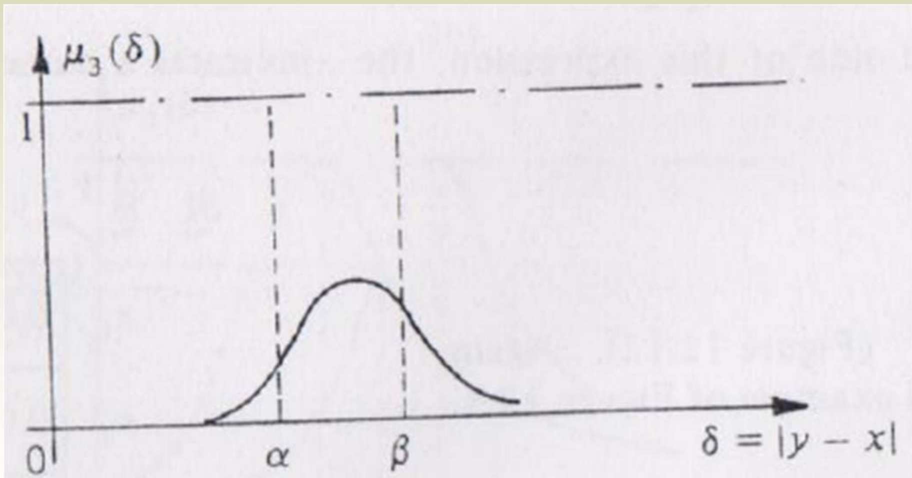
Algebraic product of two fuzzy relations

$$\mu_{\underset{\sim}{R}.\underset{\sim}{Q}}(x,y) = \mu_{\underset{\sim}{R}}(x,y).\mu_{\underset{\sim}{Q}}(x,y)$$



|y-x| is very near α

|y-x| is very near β



Distributivity property

$$\underset{\sim}{R} \cap (\underset{\sim}{Q} \cup \underset{\sim}{P}) = (\underset{\sim}{R} \cap \underset{\sim}{Q}) \cup (\underset{\sim}{R} \cap \underset{\sim}{P})$$

$$\underset{\sim}{R} \cup (\underset{\sim}{Q} \cap \underset{\sim}{P}) = (\underset{\sim}{R} \cup \underset{\sim}{Q}) \cap (\underset{\sim}{R} \cup \underset{\sim}{P})$$

$$\underset{\sim}{R}.(\underset{\sim}{Q} \cup \underset{\sim}{P}) = (\underset{\sim}{R}.\underset{\sim}{Q}) \cup (\underset{\sim}{R}.\underset{\sim}{P})$$

$$\underset{\sim}{R}.(\underset{\sim}{Q} \cap \underset{\sim}{P}) = (\underset{\sim}{R}.\underset{\sim}{Q}) \cap (\underset{\sim}{R}.\underset{\sim}{P})$$

Algebraic sum of two fuzzy relations

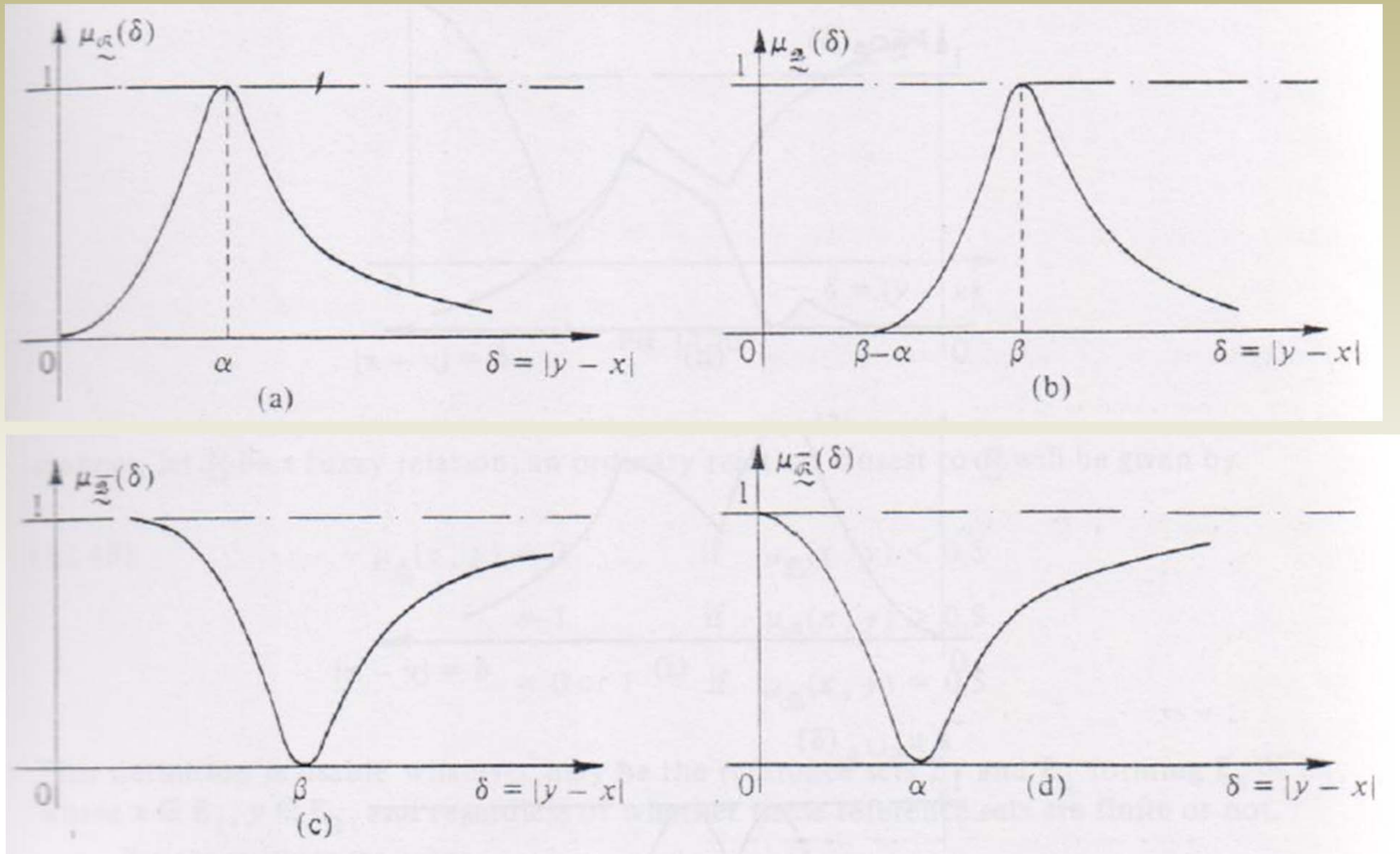
$$\mu_{\underset{\sim}{R}\hat{+}\underset{\sim}{Q}}(x,y) = \mu_{\underset{\sim}{R}}(x,y) + \mu_{\underset{\sim}{Q}}(x,y) - \mu_{\underset{\sim}{R}}(x,y).\mu_{\underset{\sim}{Q}}(x,y)$$

- The complement of a relation:

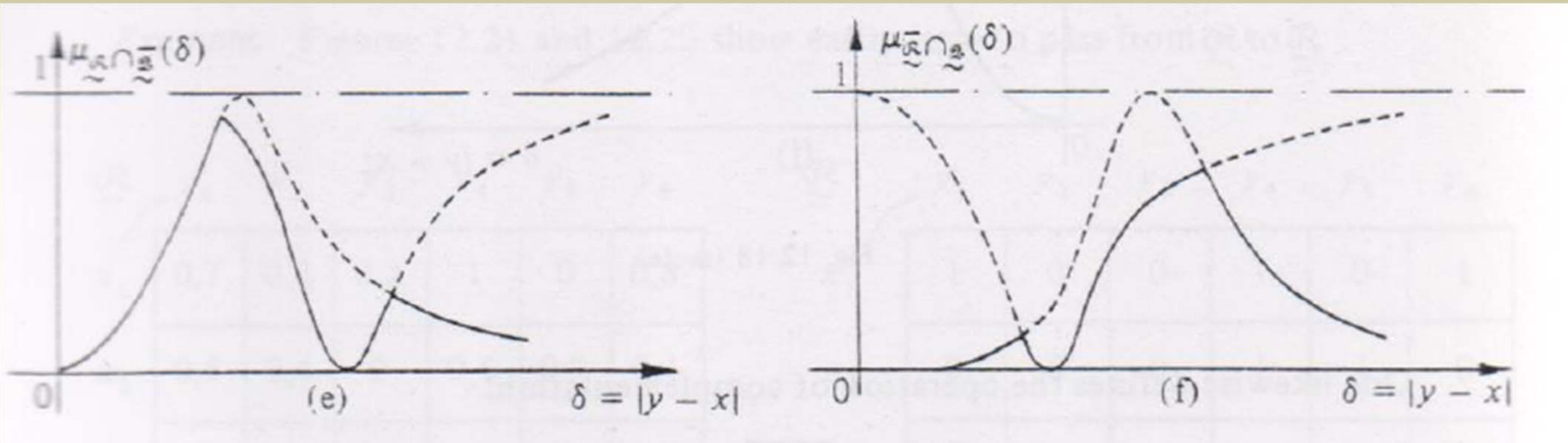
$$\mu_{\overline{\underset{\sim}{R}}}(x,y) = 1 - \mu_{\underset{\sim}{R}}(x,y)$$

Disjunctive sum of two fuzzy relations

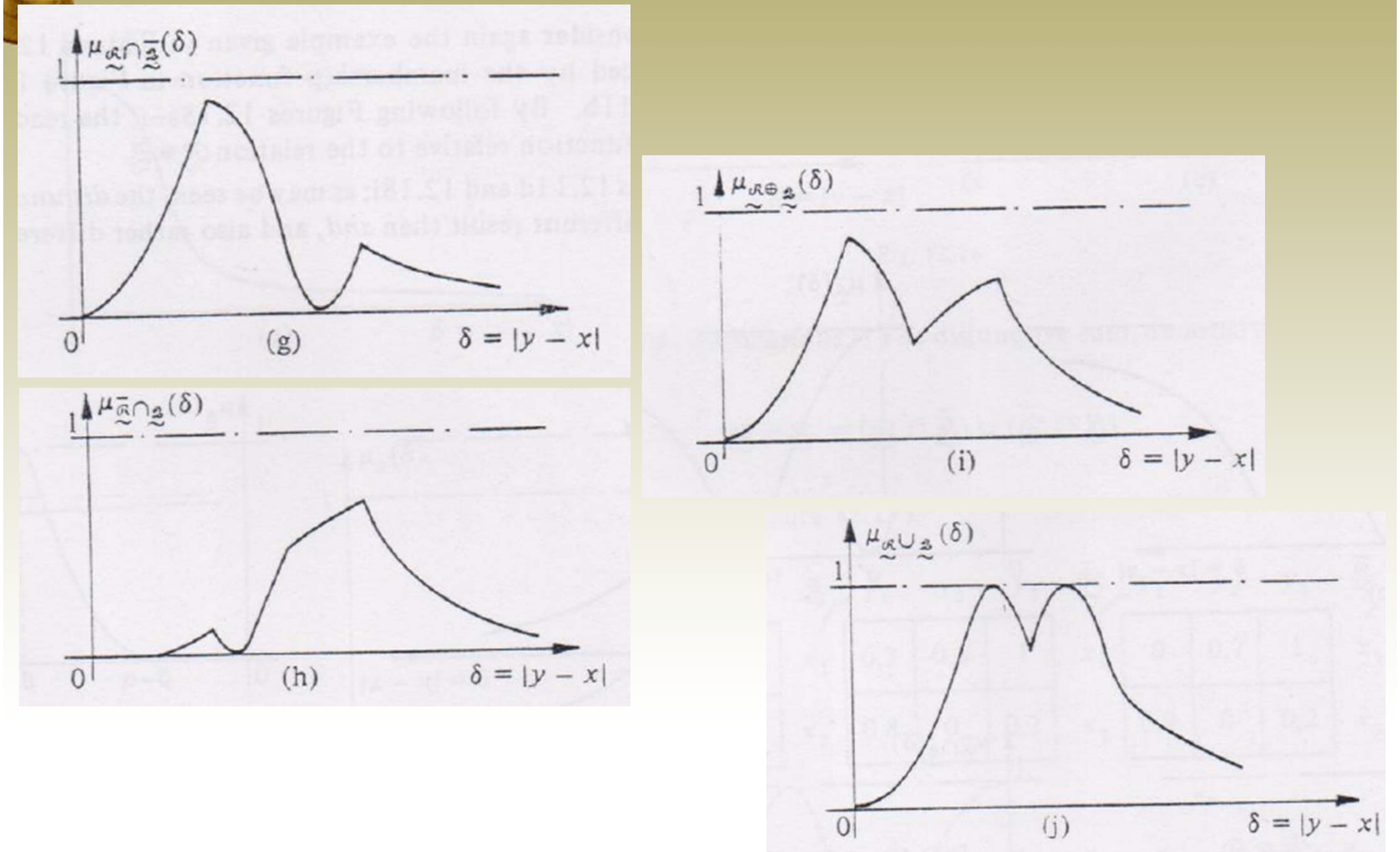
$$\underset{\sim}{R} \oplus \underset{\sim}{Q} = (\underset{\sim}{R} \cap \overline{\underset{\sim}{Q}}) \cup (\overline{\underset{\sim}{R}} \cap \underset{\sim}{Q})$$



Disjunctive sum of two fuzzy relations



Disjunctive sum of two fuzzy relations



Composition of two fuzzy relations

- <u>Max-min composition</u>

$$\mu_{\underset{\sim}{R} \circ \underset{\sim}{Q}}(x,z) = \bigvee_y [\mu_{\underset{\sim}{R}}(x,y) \wedge \mu_{\underset{\sim}{Q}}(y,z)]$$

$$= \max_y \left[\min \left\{ \mu_{\underset{\sim}{R}}(x,y), \mu_{\underset{\sim}{Q}}(y,z) \right\} \right]$$

- <u>Max-star composition</u>: we may replace the operation $\wedge$ with another, under the restriction that one uses an operation that is associative and monotone nondecreasing in each argument. Then:

$$\mu_{\underset{\sim}{R} * \underset{\sim}{Q}}(x,z) = \bigvee_y [\mu_{\underset{\sim}{R}}(x,y) * \mu_{\underset{\sim}{Q}}(y,z)]$$

- <u>Max-product composition</u>: among the max-star compositions, max-product is particularly interesting, where instead of star it uses the usual product operation

$$\mu_{\underset{\sim}{R} . \underset{\sim}{Q}}(x,z) = \bigvee_y [\mu_{\underset{\sim}{R}}(x,y) . \mu_{\underset{\sim}{Q}}(y,z)]$$

Composition of fuzzy relations: Example

$\underset{\sim}{R}$	$\mathbf{y_1}$	$\mathbf{y_2}$	$\mathbf{y_3}$	$\mathbf{y_4}$	$\mathbf{y_5}$
$\mathbf{x_1}$	0.1	0.2	0	1	0.7
$\mathbf{x_2}$	0.3	0.5	0	0.2	1
$\mathbf{x_3}$	0.8	0	1	0.4	0.3

$\underset{\sim}{Q}$	$\mathbf{z_1}$	$\mathbf{z_2}$	$\mathbf{z_3}$	$\mathbf{z_4}$
$\mathbf{y_1}$	0.9	0	0.3	0.4
$\mathbf{y_2}$	0.2	1	0.8	0
$\mathbf{y_3}$	0.8	0	0.7	1
$\mathbf{y_4}$	0.4	0.2	0.3	0
$\mathbf{y_5}$	0	1	0	0.8

Εκκινώ με (x,z)=(x_1,z_1)

$$\min(\mu_{\underset{\sim}{R}}(x_1, y_1), \mu_{\underset{\sim}{Q}}(y_1, z_1)) = \min(0.1, 0.9) = 0.1$$

$$\min(\mu_{\underset{\sim}{R}}(x_1, y_2), \mu_{\underset{\sim}{Q}}(y_2, z_1)) = \min(0.2, 0.2) = 0.2$$

$$\min(\mu_{\underset{\sim}{R}}(x_1, y_3), \mu_{\underset{\sim}{Q}}(y_3, z_1)) = \min(0, 0.8) = 0$$

$$\min(\mu_{\underset{\sim}{R}}(x_1, y_4), \mu_{\underset{\sim}{Q}}(y_4, z_1)) = \min(1, 0.4) = 0.4$$

$$\min(\mu_{\underset{\sim}{R}}(x_1, y_5), \mu_{\underset{\sim}{Q}}(y_5, z_1)) = \min(0.7, 0) = 0$$

$$\max[\min(\mu_{\underset{\sim}{R}}(x_1, y_i), \mu_{\underset{\sim}{Q}}(y_i, z_1))] = \max(0.1, 0.2, 0, 0.4, 0) = 0.4$$

$\underset{\sim}{R} \circ \underset{\sim}{Q}$	$\mathbf{z_1}$	$\mathbf{z_2}$	$\mathbf{z_3}$	$\mathbf{z_4}$
$\mathbf{x_1}$	0.4	0.7	0.3	0.7
$\mathbf{x_2}$	0.3	1	0.5	0.8
$\mathbf{x_3}$	0.8	0.3	0.7	1

Fuzzy vector-matrix multiplication

$\underset{\sim}{A}$				
	0.3	0.4	0.8	1

$\underset{\sim}{B}$			
	0.2	0.8	0.7
	0.7	0.6	0.6
	0.8	0.1	0.5
	0	0.2	0.3

$$\{\underset{\sim}{A} \circ \underset{\sim}{B}\}_j = \max_{1 \le i \le n} [\min(a_i, b_{i,j})]$$

$$\begin{aligned}
\{\underset{\sim}{A} \circ \underset{\sim}{B}\}_1 &= \max[min(0.3, 0.2), \min(0.4, 0.7), \min(0.8, 0.8), \min(1, 0)] \\
&= \min(0.2, 0.4, 0.8, 0) \\
&= 0.8 \\
\{\underset{\sim}{A} \circ \underset{\sim}{B}\}_2 &= \max(0.3, 0.4, 0.1, 0.2) \\
&= 0.4 \\
\{\underset{\sim}{A} \circ \underset{\sim}{B}\}_3 &= \max(0.3, 0.4, 0.5, 0.3) \\
&= 0.5
\end{aligned}$$

$\underset{\sim}{A} \circ \underset{\sim}{B}$			
	0.8	0.4	0.5

Exercises

Recall (from your Discrete Mathematics course) the concepts of modus ponens and modus tollens as inference rules (κανόνες συμπεράσματος):

Modus ponens (κανών αποσπάσεως): $p \wedge (p \Rightarrow q) \Rightarrow q$

Modus tollens (κανών συλλογισμού αρνητικής μορφής): $(p \Rightarrow q) \wedge !q \Rightarrow !p$

Exercise 1. Let $t : \mathbf{S} \rightarrow [0,1]$ be a continuous or "fuzzy" truth function on the set **S** of statements. Define the implication operator as the truth function $t_L(A \rightarrow B) = \min(1, 1 - t(A) + t(B))$ for statements A and B. Then prove the following generalized fuzzy *modus ponens* inference rule:

	$t_L(A \rightarrow B)$	$=$	c
	$t(A)$	$\geq$	α
Therefore	$t(B)$	$\geq$	$\max(0, \alpha + c - 1)$

Hence, if $t(A) = t_L(A \rightarrow B) = 1$, then $t(B) = 1$, which generalizes classical bivalent *modus ponens*.

Exercises

Exercise 2. Use the multivalued logic operations of the previous problem to prove the following generalized *modus tollens* inference rule:

	t_L(A→B)	=	c
	t(B)	≤	b
Therefore	t(A)	≤	min(1, 1-c+b)

Hence, if t_L(A→B)= 1 and t(B)=0, then t(A)=0, which generalizes classical bivalent *modus tollens*.

Exercise 3. Define the Gaines implication operator as:

$$t_G(A \to B) = \begin{cases} \min(1, t(B)/t(A)) & \text{if } \; t(A) > 0 \\ 1 & \text{if } \; t(A) = 0 \end{cases}$$

Use the Gaines implication operator t_G(A→B) to derive fuzzy *modus ponens* and *modus tollens* inference rules. The conclusions of the inference rules should differ from the conclusions of the inference rules in the previous two exercises.

Exercises

<u>Exercise 4</u>. Prove the following properties:

a) $\underset{\sim}{A} \cap (\underset{\sim}{A} \cup \underset{\sim}{B}) = \underset{\sim}{A}$ and $\underset{\sim}{A} \cup (\underset{\sim}{A} \cap \underset{\sim}{B}) = \underset{\sim}{A} = \underset{\sim}{A}$

b) $\emptyset \subset \underset{\sim}{A} \cap \overline{\underset{\sim}{A}} \subset \underset{\sim}{A} \cup \overline{\underset{\sim}{A}} \subset E$

c) $(\underset{\sim}{A} \cap \underset{\sim}{B}) \cup (\underset{\sim}{B} \cap \underset{\sim}{C}) \cup (\underset{\sim}{C} \cap \underset{\sim}{A}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cap (\underset{\sim}{B} \cup \underset{\sim}{C}) \cap (\underset{\sim}{C} \cup \underset{\sim}{A})$

<u>Exercise 5</u>. Simplify the expression:

$$[\underset{\sim}{A} \cap [(\underset{\sim}{B} \cap \underset{\sim}{C}) \cup (\overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{C}})]] \cup \overline{\underset{\sim}{C}}$$

Exercises

Exercise 6. Consider the reference set E =[0, α] ⊂ R. If \utilde{A} is the fuzzy subset defined by:

$$\mu_{\utilde{A}}(x) = \frac{x^2}{\alpha^2}, \quad x \in [0, \alpha]$$

then give the linear index of fuzziness of \utilde{A}.

Exercise 7. Define the *ordinary subset of level a in a fuzzy relation* exactly the same way we did for the fuzzy subsets. Then, for the fuzzy relation defined in $\Re^2$ by:

$$\mu_{\utilde{R}}(x, y) = 1 - \frac{1}{1 + x^2 + y^2}$$

calculate the (ordinary) subset of level 0.3. Provide also its geometrical interpretation.