Νευρο-Ασαφής Υπολογιστική
Neuro-Fuzzy Computing

Διδάσκων –

Δημήτριος Κατσαρός

@ Τμ. ΗΜΜΥ

Πανεπιστήμιο Θεσσαλίας

Διάλεξη 24η

Fuzzy Subset Theory

Basic set-theoretic operations

The concept of a fuzzy subset

- Let **E** be a set denumerable or not, and let x be an element of **E**. A *fuzzy subset* \utilde{A} of **E** is a set of ordered pairs:

$$\{(x|\mu_{\utilde{A}}(x))\}, \forall x \in E$$

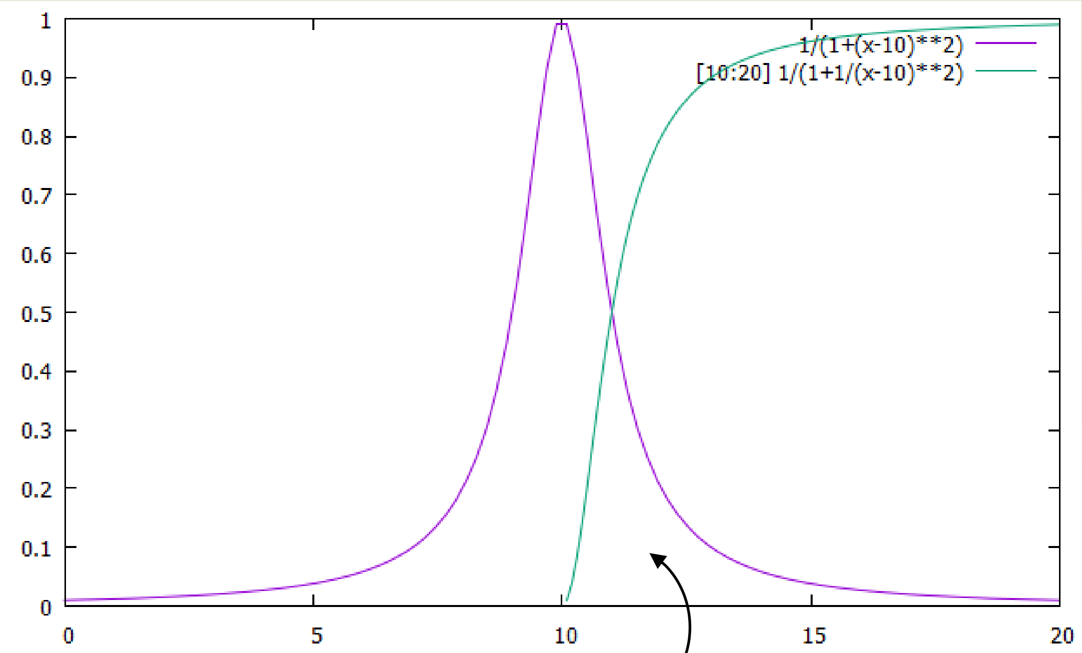
  where $\mu_{\backslash\text{utilde}\{A\}}(x)$ is a *membership characteristic function* that takes its values in a totally ordered set **M**, and which indicates the *degree* or *level of membership*. **M** will be called a *membership set*
- If **M**={0, 1}, the "fuzzy subset" \utilde{A} will be understood as an "ordinary subset"
- Cardinality of a fuzzy subset: $|\utilde{A}| = \sum_{x \in E} \mu_{\utilde{A}}(x)$
- Examples:
  1. The fuzzy subset of numbers x approximately equal to a given real number n
  2. The fuzzy subset of integers near to 0
  3. The fuzzy subset of integers very near to 0

The concept of a fuzzy subset

- Examples:
  IV. The fuzzy subset $\underset{\sim}{A}$ of real numbers close to 10
  V. The fuzzy subset $\underset{\sim}{B}$ of real numbers significantly larger than 10

$$\mu_{\underset{\sim}{A}}(x) = \frac{1}{1+(x-10)^2}$$

$$\mu_{\underset{\sim}{B}}(x) = \begin{cases} 0, & \text{if } \; x \leq 10 \\ \frac{1}{1+\frac{1}{(x-10)^2}} & x > 10 \end{cases}$$



$\underset{\sim}{A} \cap \underset{\sim}{B}$

Real numbers who are close to 10 *fuzzy-and* significantly larger than 10

Simple operations on fuzzy subsets

- **Inclusion**: We say that \utilde{A} is included in \utilde{B} if

$$\forall x \in E : \mu_{\utilde{A}}(x) \leq \mu_{\utilde{B}}(x) \quad \text{denoted as} \quad \utilde{A} \subset \utilde{B} \text{ or } \utilde{A} \utilde{\subseteq} \utilde{B}$$

$$\text{We can write:} \quad E \utilde{\subseteq} E$$

- **Strict inclusion**: If for at least one x, it holds that: $\mu_{\backslash utilde\{A\}}(x) < \mu_{\backslash utilde\{B\}}(x)$

$$\text{denoted as} \quad \utilde{A} \subset\subset \utilde{B}$$

- **Equality**:

$$\forall x \in E : \mu_{\utilde{A}}(x) = \mu_{\utilde{B}}(x) \quad \text{denoted as} \quad \utilde{A} = \utilde{B}$$

- **Complementation** (actually, it is *pseudo-complementation*):

$$\forall x \in E : \mu_{\utilde{B}}(x) = 1 - \mu_{\utilde{A}}(x) \quad \text{denoted as} \quad \utilde{B} = \overline{\utilde{A}}$$

$$\text{It holds that:} \quad \overline{(\overline{\utilde{A}})} = A$$

Simple operations on fuzzy subsets

- **Intersection**: This is the *fuzzy and*

$$\forall x \in E : \mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(x) = MIN\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\} \quad \text{denoted as} \quad \underset{\sim}{A} \cap \underset{\sim}{B}$$

- **Union**: This is the *fuzzy or/and*

$$\forall x \in E : \mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(x) = MAX\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\} \quad \text{denoted as} \quad \underset{\sim}{A} \cup \underset{\sim}{B}$$

- **Disjunctive sum**: This is the *fuzzy disjunctive or*

$$\underset{\sim}{A} \oplus \underset{\sim}{B} = (\underset{\sim}{A} \cap \overline{\underset{\sim}{B}}) \cup (\overline{\underset{\sim}{A}} \cap \underset{\sim}{B})$$

- **Difference**:

$$\underset{\sim}{A} - \underset{\sim}{B} = \underset{\sim}{A} \cap \overline{\underset{\sim}{B}}$$

Simple operations on fuzzy subsets

- **Generalized Hamming distance**:

$$d(\underset{\sim}{A},\underset{\sim}{B}) = \sum_{i=1}^{n} \left|\mu_{\underset{\sim}{A}}(x_i) - \mu_{\underset{\sim}{B}}(x_i)\right|$$

- **Euclidean distance or Quadratic distance**:

$$e(\underset{\sim}{A},\underset{\sim}{B}) = \sqrt{\sum_{i=1}^{n} \left(\mu_{\underset{\sim}{A}}(x_i) - \mu_{\underset{\sim}{B}}(x_i)\right)^2}$$

- **Generalized relative Hamming distance**:

$$\delta(\underset{\sim}{A},\underset{\sim}{B}) = \frac{d(\underset{\sim}{A},\underset{\sim}{B})}{n}$$

- **Relative Euclidean distance**:

$$\epsilon(\underset{\sim}{A},\underset{\sim}{B}) = \frac{e(\underset{\sim}{A},\underset{\sim}{B})}{\sqrt{n}}$$

Simple operations on fuzzy subsets

- **<u>The ordinary subset nearest to a fuzzy subset</u>**:

$$
\begin{aligned}
\mu_{\underline{\underset{\sim}{A}}}(x_i) &= 0 && \text{if } \mu_{\underset{\sim}{A}}(x_i) < 0.5 \\
&= 1 && \text{if } \mu_{\underset{\sim}{A}}(x_i) > 0.5 \\
&= 0 \text{ or } 1 && \text{if } \mu_{\underset{\sim}{A}}(x_i) = 0.5
\end{aligned}
$$

- **<u>Index of fuzziness</u>**:
  - Linear index of fuzziness

$$\nu(\underset{\sim}{A}) = \frac{2}{n} d(\underset{\sim}{A}, \underline{\underset{\sim}{A}})$$

  - Quadratic index of fuzziness:

$$\eta(\underset{\sim}{A}) = \frac{2}{\sqrt{n}} e(\underset{\sim}{A}, \underline{\underset{\sim}{A}})$$

Simple operations on fuzzy subsets

- **<u>Properties concerning the nearest ordinary subset</u>**:

$$\underline{\underset{\sim}{A} \cap \underset{\sim}{B}} = \underline{\underset{\sim}{A}} \cap \underline{\underset{\sim}{B}}$$

$$\underline{\underset{\sim}{A} \cup \underset{\sim}{B}} = \underline{\underset{\sim}{A}} \cup \underline{\underset{\sim}{B}}$$

$$\forall x_i \in E : |\mu_{\underset{\sim}{A}}(x_i) - \mu_{\underline{\underset{\sim}{A}}}(x_i)| = \mu_{\underset{\sim}{A} \cap \overline{\underset{\sim}{A}}}(x_i)$$

- One sometimes calls the fuzzy subset whose membership function is $2\mu_{\underset{\sim}{A} \cap \overline{\underset{\sim}{A}}}(x)$ the *vectorial indicator of fuzziness*

Simple operations on fuzzy subsets

- <u>**Evaluation of *fuzziness through entropy***</u>
- Recall the entropy of a system comprised by N states:

$$\mathcal{H}(p_1, p_2, \ldots, p_N) = -\sum_{i=1}^{N} p_i \times ln(p_i)$$

- minimum value= 0, maximum value= ln(N)
- Thus, the above equation in [0,1] becomes a measure of fuzziness:

$$\mathcal{H}(p_1, p_2, \ldots, p_N) = -\frac{1}{ln(N)} \sum_{i=1}^{N} p_i \times ln(p_i)$$

- Explanation through an example:

$$\mu_{\underset{\sim}{A}}(x_1) = 0.7, \quad \mu_{\underset{\sim}{A}}(x_2) = 0.9, \quad \mu_{\underset{\sim}{A}}(x_3) = 0.0,$$

$$\mu_{\underset{\sim}{A}}(x_4) = 0.6, \quad \mu_{\underset{\sim}{A}}(x_5) = 0.5, \quad \mu_{\underset{\sim}{A}}(x_6) = 1,$$

Simple operations on fuzzy subsets

- Putting:

$$\pi_{\underset{\sim}{A}}(x_i) = \frac{\mu_{\underset{\sim}{A}}(x_i)}{\sum_{i=1}^{6} \mu_{\underset{\sim}{A}}(x_i)}$$

- We get:

$$\pi_{\underset{\sim}{A}}(x_1) = \frac{7}{37}, \quad \pi_{\underset{\sim}{A}}(x_2) = \frac{9}{37}, \quad \pi_{\underset{\sim}{A}}(x_3) = 0.0,$$

$$\pi_{\underset{\sim}{A}}(x_4) = \frac{6}{37}, \quad \pi_{\underset{\sim}{A}}(x_5) = \frac{5}{37}, \quad \pi_{\underset{\sim}{A}}(x_6) = \frac{10}{37}$$

Therefore:

$$\mathcal{H}(\pi_1, \pi_2, \ldots, \pi_6) = -\frac{1}{ln(6)} \sum_{i=1}^{6} \pi_{\underset{\sim}{A}}(x_i) \times ln(\pi_{\underset{\sim}{A}}(x_i)) = \cdots = 0.89$$

Entropy may be used in the theory of fuzzy subsets, but it is not a good indicator

Simple operations on fuzzy subsets

- **<u>Ordinary subset of level α</u>**:
  - For α ϵ [0,1]

$$A_\alpha = \{x | \mu_{\underset{\sim}{A}}(x) \geq \alpha\}$$

  - Important property:

$$\alpha_2 \geq \alpha_1 \Rightarrow A_{a_2} \subset A_{a_1}$$

- **<u>Decomposition theorem</u>**: Any fuzzy subset \utilde{A} can be decomposed as products of ordinary subsets by the coefficients α_i

$$\underset{\sim}{A} = \max_{\alpha_i}[\alpha_1 \times A_{\alpha_1}, \alpha_2 \times A_{\alpha_2}, \ldots, \alpha_n \times A_{\alpha_n}],$$

$$0 < \alpha_i \leq 1, \quad i = 1, 2, \ldots, n$$

Decomposition theorem proof

- **Proof**: The proof is immediate:

$$\mu_{A_{\alpha_i}}(x) = 1, \text{ if } \mu_{\underset{\sim}{A}}(x) \geq \alpha_i$$

$$\mu_{A_{\alpha_i}}(x) = 0, \text{ if } \mu_{\underset{\sim}{A}}(x) < \alpha_i$$

- So, the membership function of \utilde{A} may be written:

$$\begin{aligned} \underset{\max_{\alpha_i}[\alpha_i A_{\alpha_i}]}{\mu(x)} &= \max_{\alpha_i}[\alpha_i A_{\alpha_i}] \\ &= \max_{\alpha_i \leq \mu_{\underset{\sim}{A}}(x)} [\alpha_i] \\ &= \mu_{\underset{\sim}{A}}(x) \end{aligned}$$

Decomposition theorem example

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline \multicolumn{1}{|c|}{0,2} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0,5} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{0,7} \\ \hline \end{array} = \mathrm{MAX}\left((0,2) \,.\, \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \,, \right.$$

$$(0,5) \,.\, \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \,,\; (0,7) \,.\, \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \,,$$

$$\left. (1) \,.\, \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{0} \\ \hline \end{array} \right) .$$

Set of fuzzy subsets for E and M finite

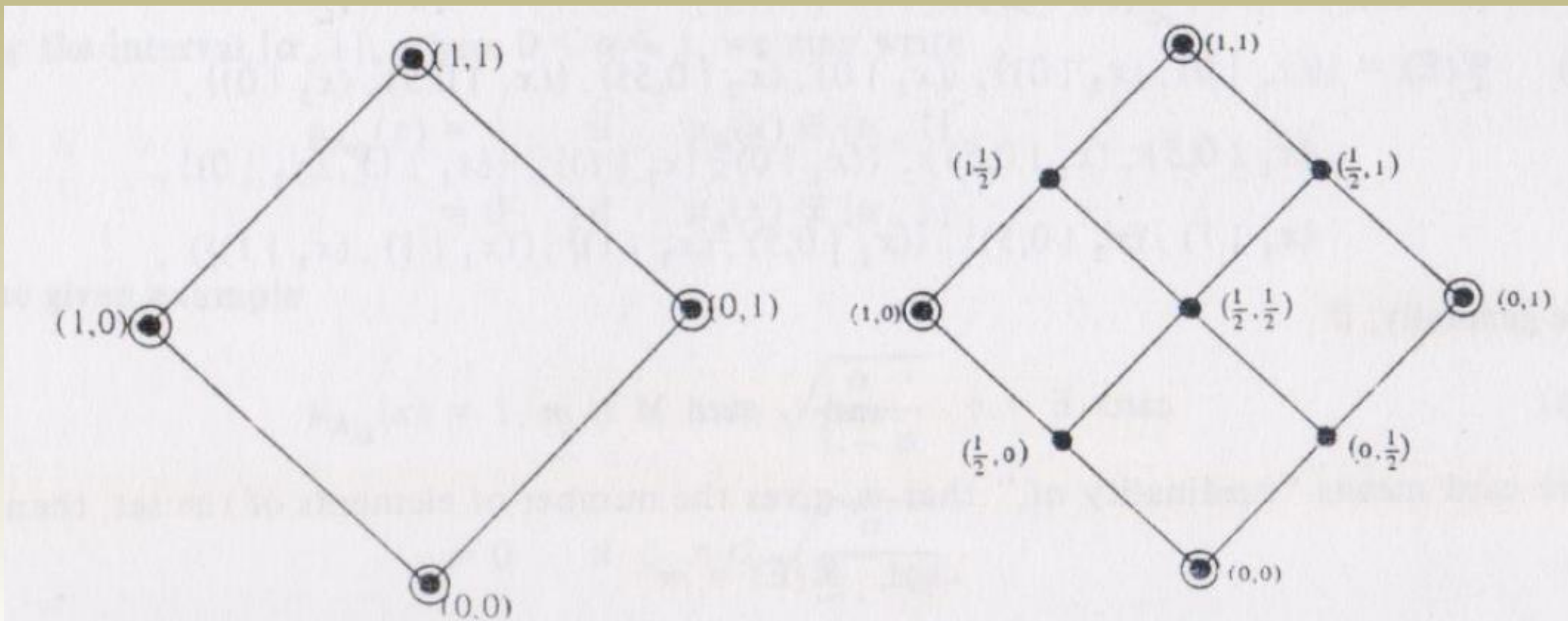
- **The powerset for a fuzzy subset**

  - If cardinality[**E**]= n and cardinality[**M**]= m, then:

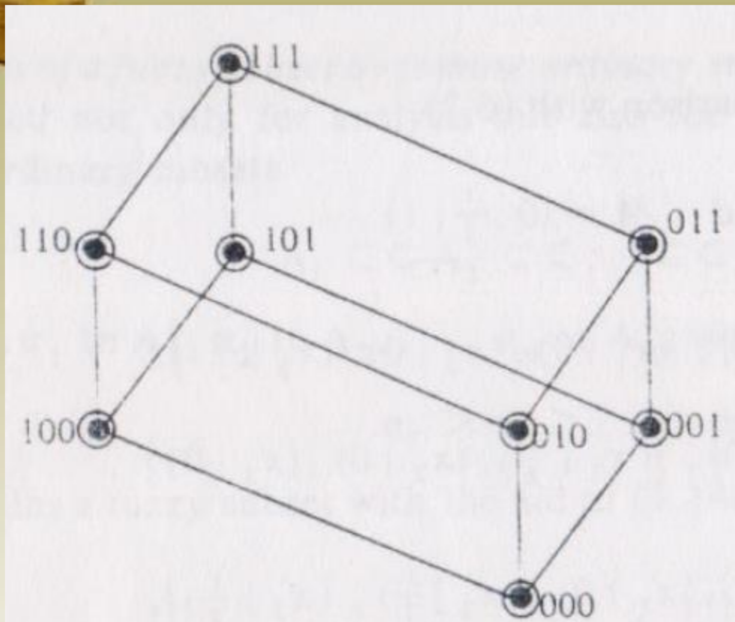
$$cardinality[\underset{\sim}{\mathcal{P}}(E)] = m^n$$

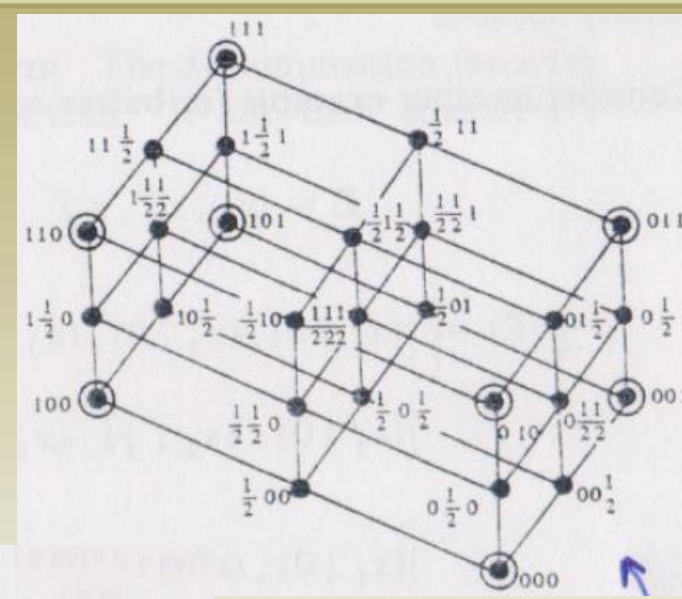
  - It is well known that the structure of a power set \mathcal{\utilde{P}}(E) of a set is a distributive and complementary lattice, that is, a *boolean lattice*. The set of fuzzy subsets \mathcal{\utilde{P}}(E), however, has the structure of a *vectorial lattice* that is distributive but not complimentary

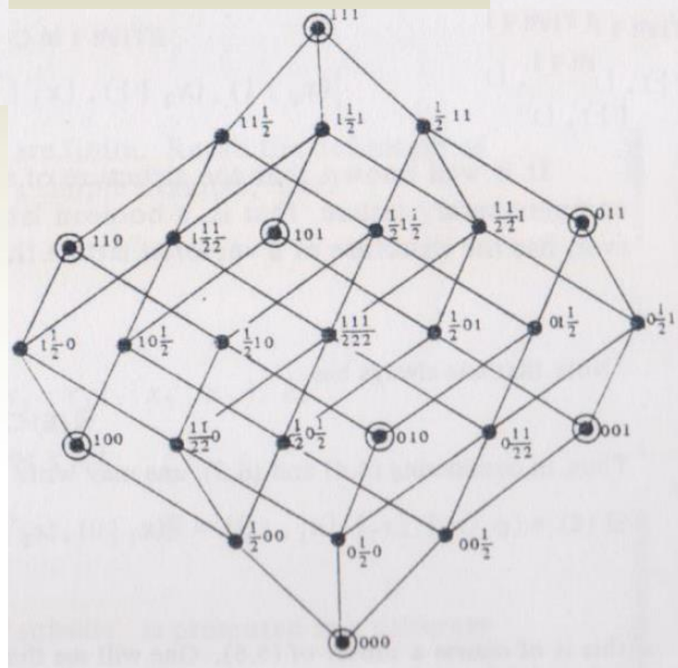
Set of fuzzy subsets for E and M finite



Set of fuzzy subsets for E and M finite







Properties of the powerset of ordinary set

$$A \cap B = B \cap A \tag{1}$$
$$A \cup B = B \cup A \tag{2}$$

commutativity properties

$$(A \cap B) \cap C = A \cap (B \cap C) \tag{3}$$
$$(A \cup B) \cup C = A \cup (B \cup C) \tag{4}$$

associativity properties

$$A \cap A = A \tag{5}$$
$$A \cup A = A \tag{6}$$

idempotence

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{7}$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{8}$$

distributivity of intersection with respect to union, and of union with respect to intersection

$$A \cap \overline{A} = \emptyset \tag{9}$$

Law of contradiction

$$A \cup \overline{A} = E \tag{10}$$

Law of excluded middle

$$A \cap \emptyset = \emptyset \tag{11}$$

absorption by $\varnothing$

$$A \cup \emptyset = A \tag{12}$$
$$A \cap E = A \tag{13}$$

identity

$$A \cup E = E \tag{14}$$

absorption by **E**

$$\overline{(\overline{A})} = A \tag{15}$$

involution

$$\overline{A \cap B} = \overline{A} \cup \overline{B} \tag{16}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B} \tag{17}$$

De Morgan’s theorems

Properties of the set of fuzzy subsets

$$\underset{\sim}{A} \cap \underset{\sim}{B} = \underset{\sim}{B} \cap \underset{\sim}{A} \quad (18)$$

$$\underset{\sim}{A} \cup \underset{\sim}{B} = \underset{\sim}{B} \cup \underset{\sim}{A} \quad (19)$$

commutativity properties

$$(\underset{\sim}{A} \cap \underset{\sim}{B}) \cap \underset{\sim}{C} = \underset{\sim}{A} \cap (\underset{\sim}{B} \cap \underset{\sim}{C}) \quad (20)$$

$$(\underset{\sim}{A} \cup \underset{\sim}{B}) \cup \underset{\sim}{C} = \underset{\sim}{A} \cup (\underset{\sim}{B} \cup \underset{\sim}{C}) \quad (21)$$

associativity properties

$$\underset{\sim}{A} \cap \underset{\sim}{A} = \underset{\sim}{A} \quad (22)$$

$$\underset{\sim}{A} \cup \underset{\sim}{A} = \underset{\sim}{A} \quad (23)$$

idempotence

$$\underset{\sim}{A} \cap (\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cup (\underset{\sim}{A} \cap \underset{\sim}{C}) \quad (24)$$

$$\underset{\sim}{A} \cup (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cap (\underset{\sim}{A} \cup \underset{\sim}{C}) \quad (25)$$

distributivity of intersection with respect to union, and of union with respect to intersection

$$\underset{\sim}{A} \cap \emptyset = \emptyset \quad (26)$$

$$\underset{\sim}{A} \cup \emptyset = \underset{\sim}{A} \quad (27)$$

where $\varnothing$ is the ordinary set, such that $\mu_{\varnothing}(x_i)=0, \forall x_i$

$$\underset{\sim}{A} \cap E = \underset{\sim}{A} \quad (28)$$

$$\underset{\sim}{A} \cup E = E \quad (29)$$

where **E** is the ordinary set, such that $\mu_{\mathbf{E}}(x_i)=1, \forall x_i$

$$\overline{(\overline{\underset{\sim}{A}})} = \underset{\sim}{A} \quad (30)$$

involution

$$\overline{\underset{\sim}{A} \cap \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}} \quad (31)$$

$$\overline{\underset{\sim}{A} \cup \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{B}} \quad (32)$$

De Morgan’s theorems

Properties of the set of fuzzy subsets

- We see that all properties (1)-(17) are satisfied except from (9) and (10)
- One may define a unique complement, but the properties (9) and (10) hold only for ordinary subsets

- Thus, we stress: All properties of an ordinary power set are found again in a power set of fuzzy subsets, except (9) and (10). Thus, we no longer have an *algebra* in the sense of the theory of ordinary sets; the structure is that of a *vector lattice*

Algebraic product and sum of two fuzzy subsets

- **Algebraic product**: **E** be an ordinary set and **M**=[0, 1]

$$\forall x \in E : \mu_{\underset{\sim}{A}.\underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \times \mu_{\underset{\sim}{B}}(x) \quad \text{denoted as} \quad \underset{\sim}{A}.\underset{\sim}{B}$$

- **Algebraic sum**:

$$\forall x \in E : \mu_{\underset{\sim}{A}\hat{+}\underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(x) - \mu_{\underset{\sim}{A}}(x) \times \mu_{\underset{\sim}{B}}(x) \quad \text{denoted as} \quad \underset{\sim}{A}\hat{+}\underset{\sim}{B}$$

- One important remark:
  - If **M**={0, 1}, i.e., we are in the case of ordinary subsets, then

$$A \cap B = A.B$$

$$A \cup B = A\hat{+}B$$

Algebraic product and sum of two fuzzy subsets

$$\underset{\sim}{A}.\underset{\sim}{B} = \underset{\sim}{B}.\underset{\sim}{A} \tag{33}$$

$$\underset{\sim}{A}\hat{+}\underset{\sim}{B} = \underset{\sim}{B}\hat{+}\underset{\sim}{A} \tag{34}$$

commutativity properties

$$(\underset{\sim}{A}.\underset{\sim}{B}).\underset{\sim}{C} = \underset{\sim}{A}.(\underset{\sim}{B}.\underset{\sim}{C}) \tag{35}$$

$$(\underset{\sim}{A}\hat{+}\underset{\sim}{B})\hat{+}\underset{\sim}{C} = \underset{\sim}{A}\hat{+}(\underset{\sim}{B}\hat{+}\underset{\sim}{C}) \tag{36}$$

associativity properties

$$\underset{\sim}{A}.\emptyset = \emptyset \tag{37}$$

$$\underset{\sim}{A}\hat{+}\emptyset = \underset{\sim}{A} \tag{38}$$

$$\underset{\sim}{A}.E = \underset{\sim}{A} \tag{39}$$

$$\underset{\sim}{A}\hat{+}E = E \tag{40}$$

$$\overline{(\overline{\underset{\sim}{A}})} = \underset{\sim}{A} \tag{41}$$

involution

$$\overline{\underset{\sim}{A}.\underset{\sim}{B}} = \overline{\underset{\sim}{A}}\hat{+}\overline{\underset{\sim}{B}} \tag{42}$$

$$\overline{\underset{\sim}{A}\hat{+}\underset{\sim}{B}} = \overline{\underset{\sim}{A}}.\overline{\underset{\sim}{B}} \tag{43}$$

De Morgan's theorems

- Only the above properties are verified. Idempotence [(5) and (6)], distributivity [(7) and (8)], and of course (9) and (10) are not satisfied

Algebraic product and sum of two fuzzy subsets

- Note that $\cup$ is not distributive with respect to . or $\hat{+}$, and likewise $\cap$, but on the other hand one has:

$$\underset{\sim}{A}.(\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A}.\underset{\sim}{B}) \cap (\underset{\sim}{A}.\underset{\sim}{C}) \tag{44}$$

$$\underset{\sim}{A}.(\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A}.\underset{\sim}{B}) \cup (\underset{\sim}{A}.\underset{\sim}{C}) \tag{45}$$

$$\underset{\sim}{A}\hat{+}(\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A}\hat{+}\underset{\sim}{B}) \cap (\underset{\sim}{A}\hat{+}\underset{\sim}{C}) \tag{46}$$

$$\underset{\sim}{A}\hat{+}(\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A}\hat{+}\underset{\sim}{B}) \cup (\underset{\sim}{A}\hat{+}\underset{\sim}{C}) \tag{47}$$

Algebraic product and sum of two fuzzy subsets

- Let us prove (42):
  - Suppose that $\mu_A(x)=a$ and $\mu_B(x)=b$
  - The left part gives: 1-ab
  - The right part gives: (1-a)+(1-b)-(1-a)(1-b)= 1-a+1-b-1-ab+a+b= 1-ab
  - Thus, the two parts are alike
- Let us disprove that distributivity holds, i.e., that

$$\underset{\sim}{A}.(\underset{\sim}{B}\hat{+}\underset{\sim}{C}) \neq (\underset{\sim}{A}.\underset{\sim}{B})\hat{+}(\underset{\sim}{A}.\underset{\sim}{C})$$

  - The left part gives: a (b+c-bc)= ab +ac -abc
  - The right part gives: ab + ac –abac= ab +ac –a^2bc

Fuzzy relation

- Example 1
  - $E_1 = \{x_1, x_2, x_3\}$
  - $E_2 = \{y_1, y_2, y_3, y_4, y_5\}$
  - M= [0, 1]

	y_1	y_2	y_3	y_4	y_5
x_1	0	0	0,1	0,3	1
x_2	0	0,8	0	0	1
x_3	0,4	0,4	0,5	0	0,2

- Example 2
  - $E_1 = E_2 = R$
  - Η σχέση: y << x is a fuzzy relation

$$\mu_{R^2}(x, y) = \begin{cases} 0, & \text{if } \; y \geq x \\ \frac{1}{1+\frac{1}{(x-y)^2}} & y < x \end{cases}$$

Projection of a fuzzy relation

- *First projection* of \utilde{*R*}
- *Second projection* of \utilde{*R*}
- The second projection of the first projection (or vice versa) will be called the *global projection*

$$\mu^{(1)}_{\underset{\sim}{R}}(x) = \bigvee_{y} \mu_{\underset{\sim}{R}}(x, y)$$

$$\mu^{(2)}_{\underset{\sim}{R}}(y) = \bigvee_{x} \mu_{\underset{\sim}{R}}(x, y)$$

$$h(\underset{\sim}{R}) = \bigvee_{x} \bigvee_{y} \mu_{\underset{\sim}{R}}(x, y)$$

$$= \bigvee_{y} \bigvee_{x} \mu_{\underset{\sim}{R}}(x, y)$$

$\underset{\sim}{R}$	y_1	y_2	y_3	y_4	1^{st} proj.
x_1	0,1	0,2	1	0,3	1
x_2	0,6	0,8	0	0,1	0,8
x_3	0	1	0,3	0,6	1
x_4	0,8	0,1	1	0	1
x_5	0,9	0,7	0	0,5	0,9
x_6	0,9	0	0,3	0,7	0,9
2^{nd} proj.	0,9	1	1	0,7	1 (global projection)

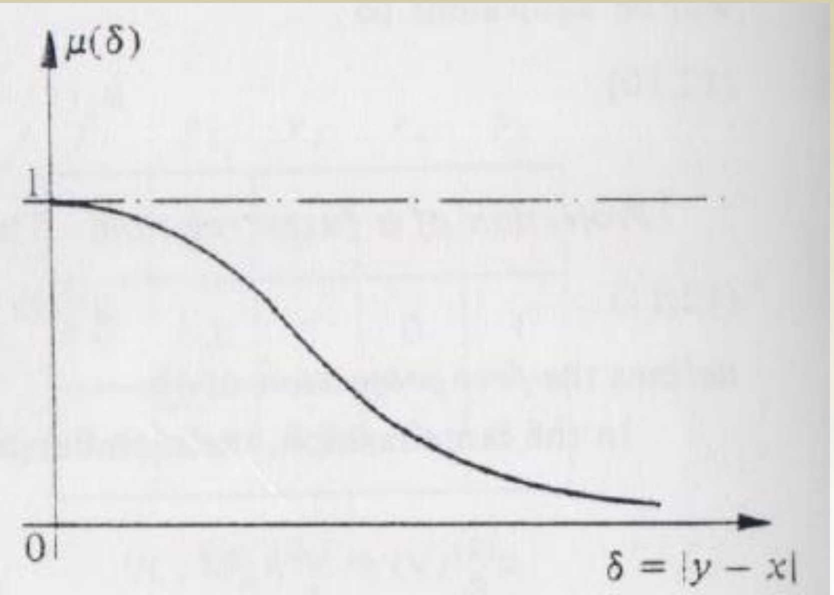
Projection of a fuzzy relation: Example 2

- x and y are very near to one another:

$$\mu_{\underset{\sim}{R}}(x,y) = e^{-k(y-x)^2}, \quad k > 1$$

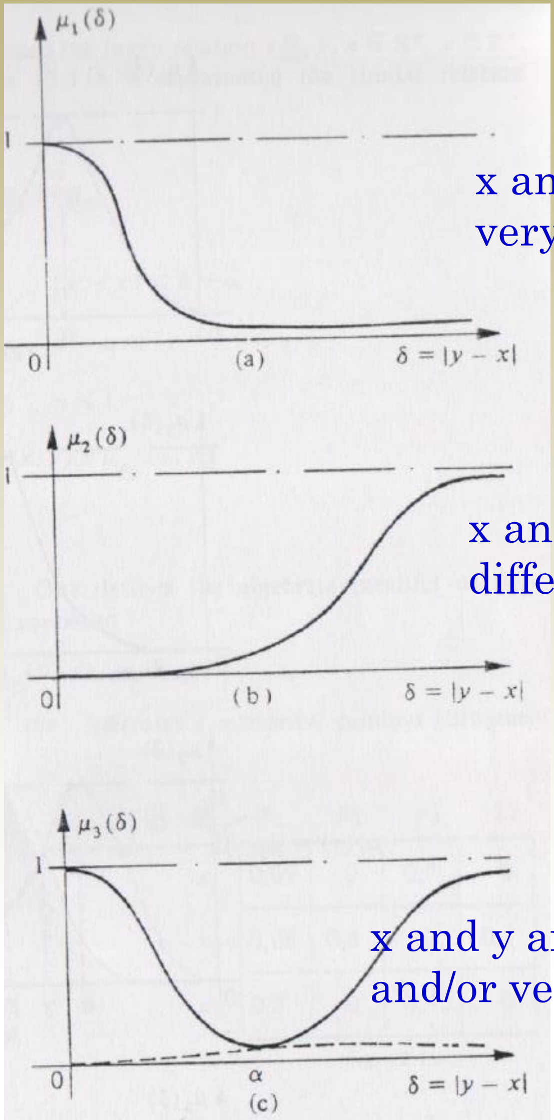
- For a fixed value x_0:

$$\begin{aligned}
\mu^{(1)}_{\underset{\sim}{R}}(x_0) &= \bigvee_y \mu_{\underset{\sim}{R}}(x_0, y) \\
&= \bigvee_y e^{-k(y-x_0)^2} \\
&= e^{-k(y-x_0)^2} \quad \text{for } y = x_0 \\
&= 1
\end{aligned}$$



Union of two relations

$$\mu_{\underset{\sim}{R}\cup\underset{\sim}{Q}}(x,y) = \mu_{\underset{\sim}{R}}(x,y) \vee \mu_{\underset{\sim}{Q}}(x,y)$$
$$= \max[\mu_{\underset{\sim}{R}}(x,y), \mu_{\underset{\sim}{Q}}(x,y)]$$



x and y are very near

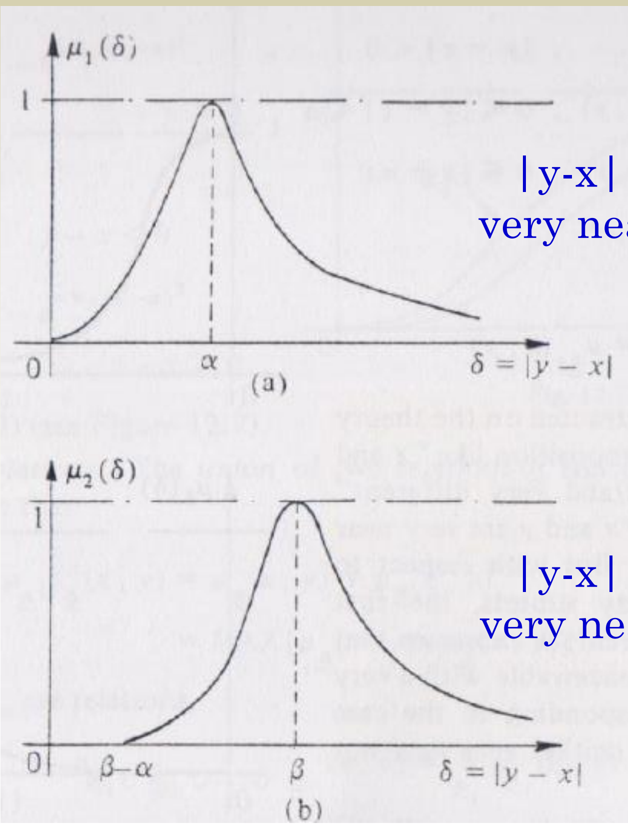
x and y are very different

x and y are very near and/or very different

Intersection of two relations

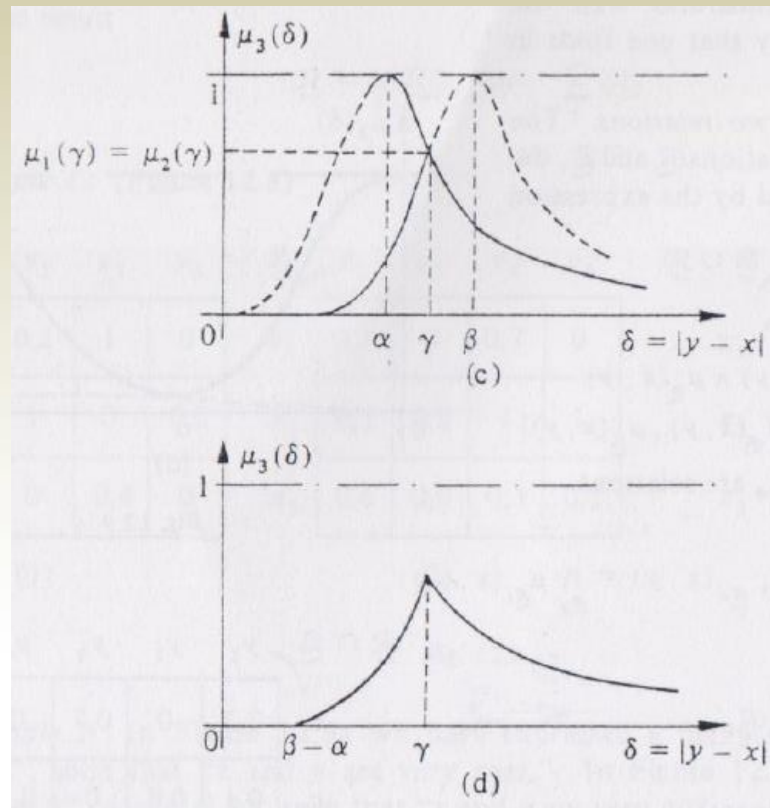
$$\mu_{\underset{\sim}{R} \cap \underset{\sim}{Q}}(x, y) = \mu_{\underset{\sim}{R}}(x, y) \wedge \mu_{\underset{\sim}{Q}}(x, y)$$

$$= \min[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{Q}}(x, y)]$$



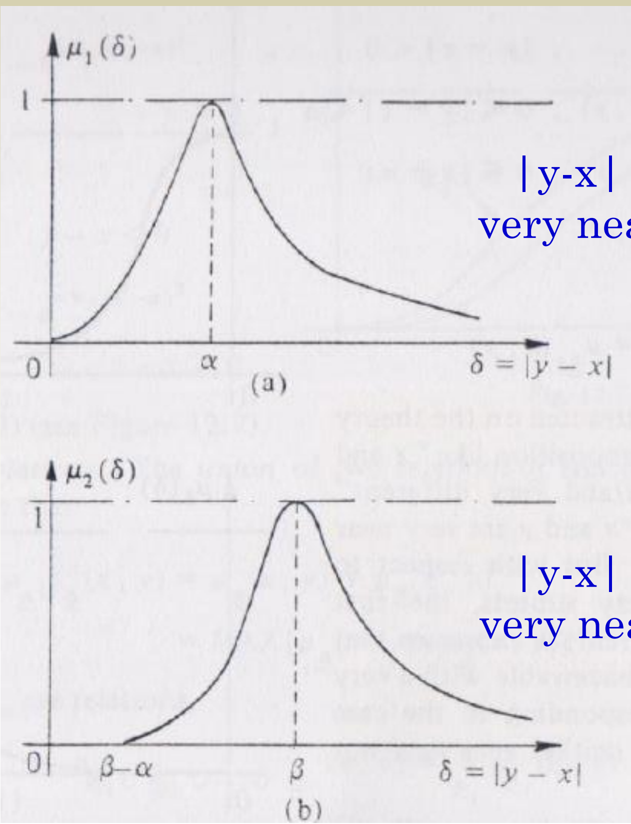
|y-x| is very near α

|y-x| is very near β



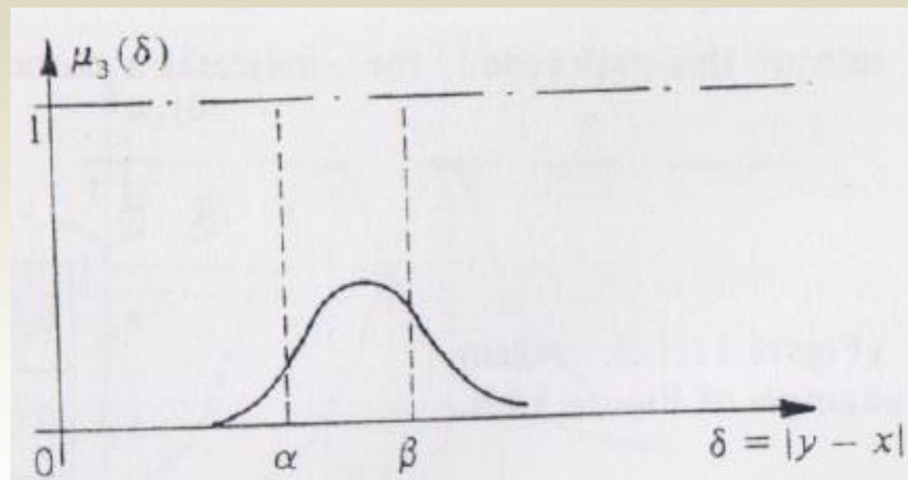
Algebraic product of two relations

$$\mu_{\underset{\sim}{R}.\underset{\sim}{Q}}(x,y) = \mu_{\underset{\sim}{R}}(x,y).\mu_{\underset{\sim}{Q}}(x,y)$$



|y-x| is
very near α

|y-x| is
very near β



Distributivity property

$$\underset{\sim}{R} \cap (\underset{\sim}{Q} \cup \underset{\sim}{P}) = (\underset{\sim}{R} \cap \underset{\sim}{Q}) \cup (\underset{\sim}{R} \cap \underset{\sim}{P})$$

$$\underset{\sim}{R} \cup (\underset{\sim}{Q} \cap \underset{\sim}{P}) = (\underset{\sim}{R} \cup \underset{\sim}{Q}) \cap (\underset{\sim}{R} \cup \underset{\sim}{P})$$

$$\underset{\sim}{R} . (\underset{\sim}{Q} \cup \underset{\sim}{P}) = (\underset{\sim}{R} . \underset{\sim}{Q}) \cup (\underset{\sim}{R} . \underset{\sim}{P})$$

$$\underset{\sim}{R} . (\underset{\sim}{Q} \cap \underset{\sim}{P}) = (\underset{\sim}{R} . \underset{\sim}{Q}) \cap (\underset{\sim}{R} . \underset{\sim}{P})$$

Algebraic sum of two relations

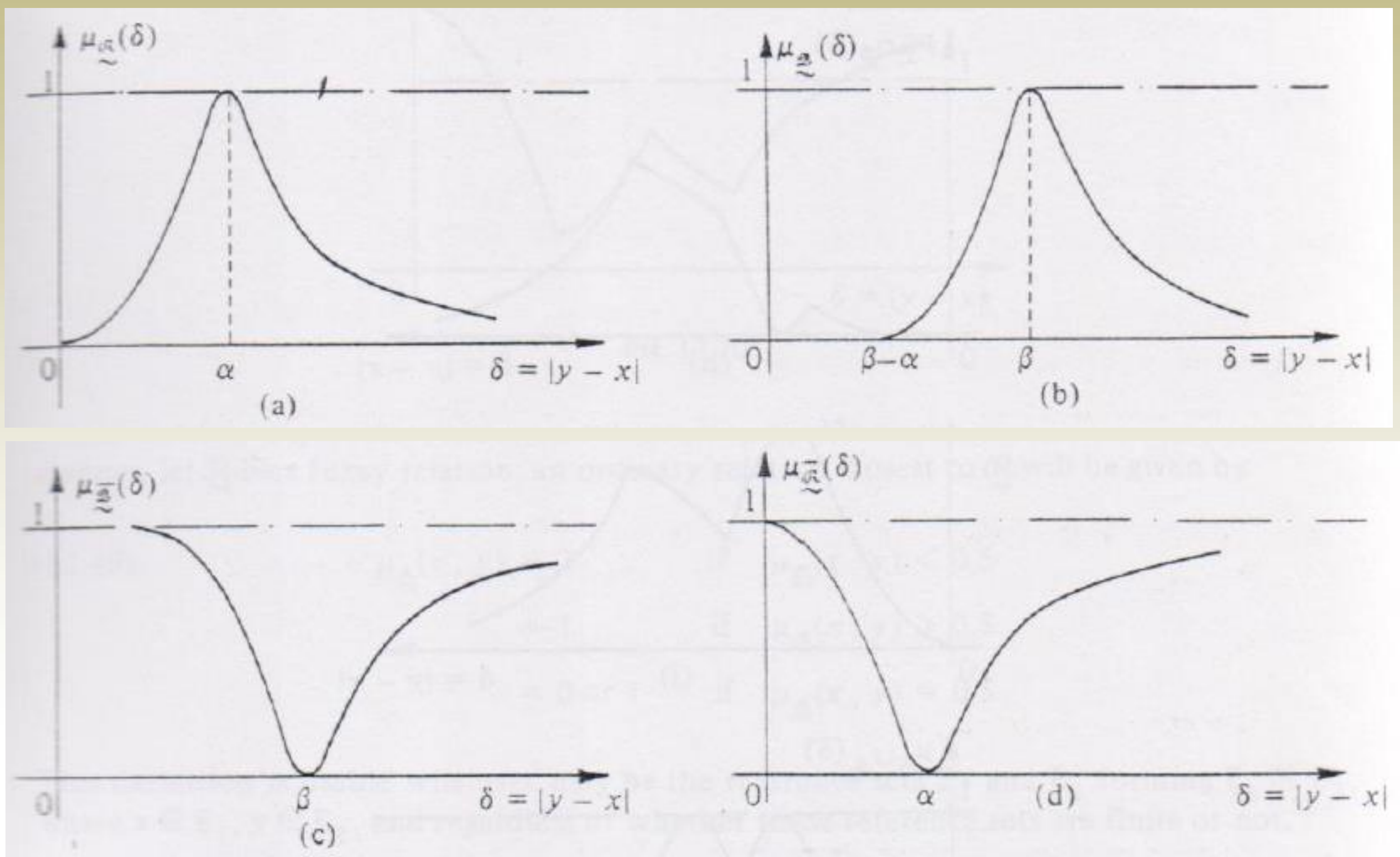
$$\mu_{\underset{\sim}{R}\hat{+}\underset{\sim}{Q}}(x,y) = \mu_{\underset{\sim}{R}}(x,y) + \mu_{\underset{\sim}{Q}}(x,y) - \mu_{\underset{\sim}{R}}(x,y).\mu_{\underset{\sim}{Q}}(x,y)$$

- The complement of a relation:

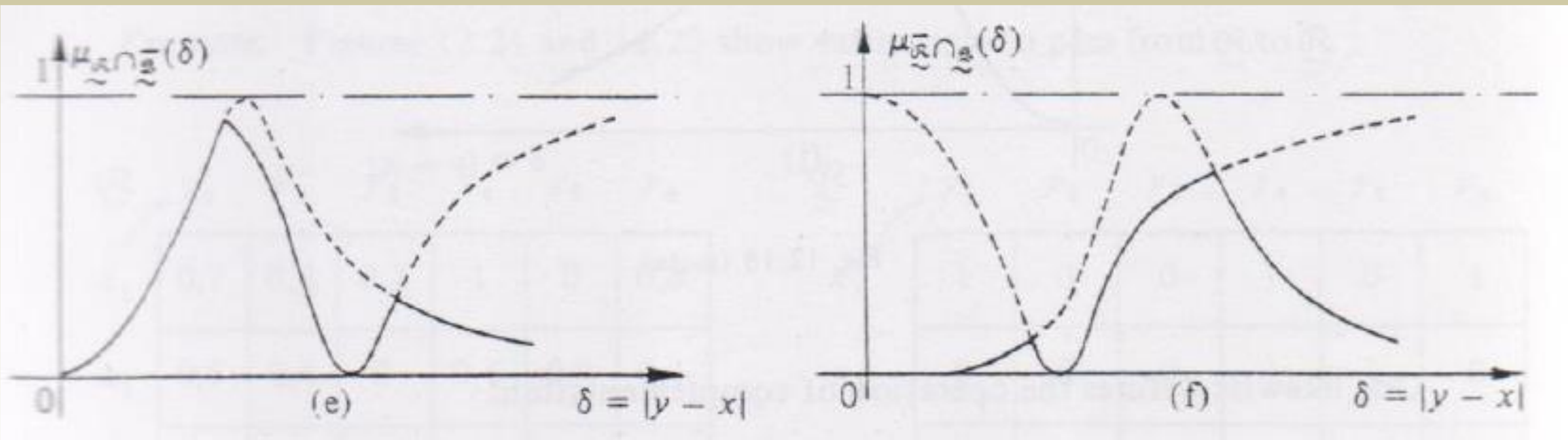
$$\mu_{\overline{\underset{\sim}{R}}}(x,y) = 1 - \mu_{\underset{\sim}{R}}(x,y)$$

Disjunctive sum of two relations

$$\underset{\sim}{R} \oplus \underset{\sim}{Q} = (\underset{\sim}{R} \cap \overline{\underset{\sim}{Q}}) \cup (\overline{\underset{\sim}{R}} \cap \underset{\sim}{Q})$$



Disjunctive sum of two relations



Disjunctive sum of two relations