S-38.143 Queueing Theory, Fall 2004
Exercises

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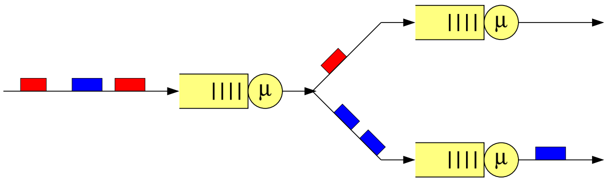

Contents

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Glossary

ACK	a signal acknowledging a succesful transfer, 25
DSL	Digital subscriber line, 25
MTU	Maximum transmissible unit, largest possible IP datagram that can be sent, 25
MVA	Mean value analysis, 33
PASTA	Poisson Arrivals See Time Averages, 18
TCP	Transmission control protocol, 25

EX 1: Discrete Random Variables

1. **You are given a possibly biased coin, i.e. $p = \mathrm{P}\{\text{heads}\}$ is not necessarily 0.5. Find an experimental arrangement where using the given coin you are able to choose randomly between two alternatives A and B with equal probabilities, $\mathrm{P}\{A\} = \mathrm{P}\{B\} = 0.5$. (Hint: consider, for example, the possible results of two consecutive trials)**

   Solution:

   Denote heads by 0 and tails by 1. Consider an experiment consisting of two throws, the results of consecutive throws are naturally independent. The possible outcomes and the respective probabilities are,

$$\begin{aligned}
\mathrm{P}\{00\} &= p^2 \\
\mathrm{P}\{01\} &= p(1-p) \\
\mathrm{P}\{10\} &= (1-p)p \\
\mathrm{P}\{11\} &= (1-p)^2
\end{aligned}$$

   In other words, the outcomes 01 and 10 are equally likely. Thus, we can choose that outcome 01 means we choose A and outcome 10 means we choose B. In all other cases we can repeat the experiment.

2. **A connection consists of 4 unreliable consecutive links. On each link the probability that a transmitted bit (0 or 1) is received correctly is 90% and with probability of 10% the received bit has flipped into the other one. What is the probability that a transmitted bit is received correctly at the other end of the connection?**

   Solution:

   A bit is received correctly if 1) it does not get changed at all, or 2) the number of changes is a even number. As we assume that the changing probability on each link is independent, the total number of changes obeys binomial distribution with parameter 0.1. Hence,

$$\begin{aligned}
\mathrm{P}\{\text{bit ok}\} &= \sum_{i=0}^{2} \mathrm{P}\{2i \text{ changes}\} = \sum_{i=0}^{2} \binom{4}{2i} p^{2i}(1-p)^{4-2i} \\
&= \binom{4}{0} \cdot 0.1^0 \cdot 0.9^4 + \binom{4}{2} \cdot 0.1^2 \cdot 0.9^2 + \binom{4}{4} \cdot 0.1^4 \cdot 0.9^0 \\
&= 0.6561 + 0.0486 + 0.0001 = 0.7048.
\end{aligned}$$

3. **Tower property.**

   a) Prove that for the conditional covariance it holds that,

$$\mathbf{Cov}[X, Y|Z] = \mathbf{E}[XY|Z] - \mathbf{E}[X|Z]\,\mathbf{E}[Y|Z]\,.$$

   b) Prove the tower property of covariance,

$$\mathbf{Cov}[X, Y] = \mathbf{E}[\mathbf{Cov}[X, Y|Z]] + \mathbf{Cov}[\mathbf{E}[X|Z]\,, \mathbf{E}[Y|Z]].$$

   Hint: use the tower property of expectation, $\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X|Y]]$.

Conditional expectation:

i) Conditional expectation $\mathrm{E}[X|Y]$ is random variable,

$$\mathrm{E}[X|Y] = h(Y),$$

for some function h.

ii) Conditional expectation is linear,

$$\mathrm{E}[X_1 + X_2|Y] = \mathrm{E}[X_1|Y] + \mathrm{E}[X_2|Y].$$

iii) Constant terms can be taken outside from conditional expectation,

$$\mathrm{E}[g(Y)|Y] = g(Y) \qquad \text{and} \qquad \mathrm{E}[g(Y)X|Y] = g(Y)\mathrm{E}[X|Y]$$

iv) Tower property of expectation:

$$\begin{aligned} \mathrm{E}[X] &= \mathrm{E}[\mathrm{E}[X|Y]] \\ \mathrm{V}[X] &= \mathrm{E}[\mathrm{V}[X|Y]] + \mathrm{V}[\mathrm{E}[X|Y]] \end{aligned} \tag{1.1}$$

Table 1: Conditional expectation.

Solution:

a) Using the results presented in Table 1 gives,

$$\begin{aligned} \mathrm{Cov}[X,Y|Z] &\overset{\text{def}}{=} \mathrm{E}[(X - \mathrm{E}[X|Z]) \cdot (Y - \mathrm{E}[Y|Z])|Z] \\ &= \mathrm{E}[XY - X\mathrm{E}[Y|Z] - \mathrm{E}[X|Z]Y + \mathrm{E}[X|Z]\mathrm{E}[Y|Z]|Z] \\ &= \mathrm{E}[XY|Z] - \mathrm{E}[X|Z]\mathrm{E}[Y|Z] - \mathrm{E}[X|Z]\mathrm{E}[Y|Z] + \mathrm{E}[X|Z]\mathrm{E}[Y|Z] \\ &= \mathrm{E}[XY|Z] - \mathrm{E}[X|Z]\mathrm{E}[Y|Z]. \end{aligned}$$

b)

$$\begin{aligned} \mathrm{Cov}[X,Y] &= \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y] \\ &= \mathrm{E}[\mathrm{E}[XY|Z]] - \mathrm{E}[\mathrm{E}[X|Z]] \cdot \mathrm{E}[\mathrm{E}[Y|Z]] \end{aligned}$$

and adding and subtracting term $\mathrm{E}[\mathrm{E}[X|Z] \cdot \mathrm{E}[Y|Z]]$ gives,

$$\begin{aligned} &= \mathrm{E}[\mathrm{E}[XY|Z]] - \mathrm{E}[\mathrm{E}[X|Z] \cdot \mathrm{E}[Y|Z]] \\ &\quad + \mathrm{E}[\mathrm{E}[X|Z] \cdot \mathrm{E}[Y|Z]] - \mathrm{E}[\mathrm{E}[X|Z]] \cdot \mathrm{E}[\mathrm{E}[Y|Z]] \\ &= \mathrm{E}[\mathrm{E}[XY|Z] - \mathrm{E}[X|Z] \cdot \mathrm{E}[Y|Z]] \\ &\quad + \mathrm{E}[\mathrm{E}[X|Z] \cdot \mathrm{E}[Y|Z]] - \mathrm{E}[\mathrm{E}[X|Z]] \cdot \mathrm{E}[\mathrm{E}[Y|Z]] \\ &= \mathrm{E}[\mathrm{Cov}[X,Y|Z]] + \mathrm{Cov}[\mathrm{E}[X|Z], \mathrm{E}[Y|Z]]. \end{aligned}$$

Note that $\mathrm{Cov}[X,X] = \mathrm{V}[X]$ and the tower property of variance is a special case of b).

4. **Consider binomially distributed random variables.**

   a) Let $N \sim \mathrm{Bin}(n,p)$ denote the size of a population, for which we do random selection into k subsets with probabilities q_i, $\sum_i q_i = 1$. Let N_i denote the size of subset i. Show that each N_i obeys binomial distribution.

   b) Let N_i^*, $i = 1, \ldots, k$, be a set of independent and binomially distributed random variables, $N_i^* \sim \mathrm{Bin}(n, pq_i)$. Determine the generating function of sum $N^* = N_1^* + \ldots + N_k^*$.

   c) Why is $N^* \nsim N$ generally?

Solution:

a) Generally, when n independent trials are performed where the probability of success on each trial is p, the total number of succesful trials, denoted by N, obeys binomial distribution, $N \sim \mathrm{Bin}(n, p)$.

Consider the size of set j denoted by N_j. According to Fig. 1 an individual ends up in set j, if two consecutive trials are succesful: first with probability of p and then with probability of q_j. In other words, with probability of pq_j both trials are succesful and hence $N_j \sim \mathrm{Bin}(n, pq_j)$.

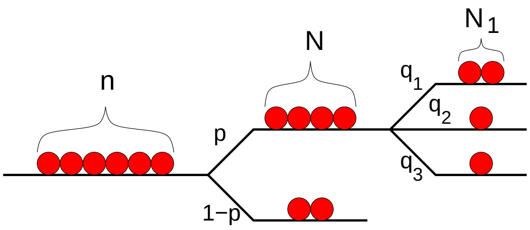


Figure 1: The no. of succesful trials obeys binomial distribution.

b) The generating function of binomially distributed random variable is

$$\mathcal{G}(z) = \sum_{i=0}^{n} \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i} \cdot z^i = (pz + (1-p))^n,$$

and thus the generating function of N_i^* is

$$\mathcal{G}_i^*(z) = (pq_i z + (1 - pq_i))^n.$$

The generating function of the sum of random variables is the product of generating functions, i.e.

$$\mathcal{G}^*(z) = \mathcal{G}_1^*(z) \cdot \ldots \cdot \mathcal{G}_k^*(z) = [\,(pq_1 z + (1-pq_1)) \cdot \ldots \cdot (pq_k z + (1-pq_k))]^n.$$

Note that the sum is binomially distributed only if $q_1 = \ldots = q_k = 1/k$, for which we have $N^* \sim \mathrm{Bin}(kn, p/k)$.

c) The difference is simply due to the fact that in a) the random variables N_i are not independent. For example, when $q_j = 1/k$ for all j, based on above we have $N^* \sim \mathrm{Bin}(kn, p/k)$, when clearly $N \nsim N^*$ (means are equal, but, e.g. the variancec are different).

5. **Prove (without using the generating function), that the sum of two Poisson random variables, $N_1 \sim$ Poisson(a_1) and $N_2 \sim$ Poisson(a_2), is also Poisson distributed: $(N_1 + N_2) \sim$ Poisson($a_1 + a_2$). Prove the same result with the aid of generating functions.**

Solution:

Poisson-distribution: $P\{N_i = n\} = \frac{(a_i)^n}{n!} \cdot e^{-a_i}$.

$$\begin{aligned}
\mathrm{P}\{N = n\} &= \mathrm{P}\{N_1 + N_2 = n\} = \sum_{j=0}^{n} \mathrm{P}\{N_1 = j\} \cdot \mathrm{P}\{N_2 = n - j\} \\
&= \sum_{j=0}^{n} \frac{(a_1)^j}{j!} \cdot e^{-a_1} \cdot \frac{(a_2)^{n-j}}{(n-j)!} \cdot e^{-a_2} = \frac{e^{-(a_1+a_2)}}{n!} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \cdot (a_1)^j (a_2)^{n-j} \\
&= \frac{(a_1 + a_2)^n \cdot}{n!} \cdot e^{-(a_1+a_2)}, \qquad \text{(binomial theorem)}
\end{aligned}$$

thus the sum $N = N_1 + N_2$ is Poisson($a_1 + a_2$).

On the other hand, let $X \sim \mathrm{Poisson}(a)$. Then the generating function of the random variable X is

$$\mathcal{G}X(z) = E[z^X] = \sum_{j=0}^{\infty} z^j \frac{a^j}{j!} e^{-a} = e^{-a} \sum_{j=0}^{\infty} \frac{(az)^j}{j!} = e^{-a} e^{az} = e^{(z-1)a}.$$

Let $N(z)$ denote the generating function of sum $N_1 + N_2$, for which we obtain

$$N(z) = N_1(z) \cdot N_2(z) = e^{(z-1)a_1} \cdot e^{(z-1)a_2} = e^{(z-1)(a_1+a_2)}.$$

Hence, N obeys Poisson distribution with parameter $a_1 + a_2$.

EX 2: Continuous Distributions, Stochastic Processes

1. **Customers arrive in groups at a queueing system. The number of customers in each group is either 1 or 3 with equal probabilities. During a certain time period the number of arriving groups K obeys Poisson distribution with mean $a = 4$.**

   a) Determine first the generating function of the number of arriving customers N during the time period, and then derive the mean $\mathrm{E}[N]$ and the variance $\mathrm{V}[N]$ using the generating function.

   b) Derive the mean and variance using the chain rule (tower property) of expectation by conditioning on the number of arriving groups K.

Solution:

a) Let A denote the number of customers in one group,

$$\mathrm{P}\{A=1\} = \mathrm{P}\{A=3\} = 1/2.$$

The total number of arriving customers is a random sum

$$N = A_1 + \ldots + A_K,$$

where $K \sim \mathrm{Poisson}(4)$. Thus, the generating functions are

$$\begin{aligned}
\mathcal{G}_A(z) &= \frac{1}{2}z + \frac{1}{2}z^3 \\
\mathcal{G}_K(z) &= \sum_i \frac{a^i}{i!}e^{-a} \cdot z^i = e^{(z-1)a}, \\
\mathcal{G}_N(z) &= \mathcal{G}_K(\mathcal{G}_A(z)) = \underline{e^{2(z^3+z-2)}}.
\end{aligned}$$

The mean value is

$$\begin{aligned}
\mathcal{G}'_N(z) &= e^{2(z^3+z-2)} \cdot 2 \cdot (3z^2+1), \\
\mathrm{E}[N] &= \mathcal{G}'_N(1) = \underline{8},
\end{aligned}$$

and similarly the variance of N:

$$\begin{aligned}
\frac{d}{dz}\, z\mathcal{G}'_N(z) &= 2 \cdot \frac{d}{dz}\, e^{2(z^3+z-2)} \cdot z(3z^2+1) \\
&= 4e^{2(z^3+z-2)}(3z^2+1)^2 z + 2e^{2(z^3+z-2)}(9z^2+1) \\
&= 2 \cdot e^{2(z^3+z-2)} \cdot (2z(1+3z^2)^2 + (9z^2+1)) \\
\mathrm{E}\left[N^2\right] &= \left(\frac{d}{dz}\, z\mathcal{G}'_N(z)\right)_{|z=1} = 2 \cdot (2 \cdot 4^2 + 10) = 84, \\
\mathrm{V}[N] &= \mathrm{E}\left[N^2\right] - \mathrm{E}[N]^2 = 84 - 8^2 = \underline{20}.
\end{aligned}$$

b) The variances of A and K are

$$\begin{aligned}
\mathrm{V}[A] &= (1/2) \cdot ((1-2)^2 + (3-2)^2) = 1, \\
\mathrm{V}[K] &= a = 4,
\end{aligned}$$

and hence,

$$\begin{aligned}
\mathrm{E}[N] &= \mathrm{E}[\mathrm{E}[N|K]] = \mathrm{E}[K \cdot 2] = 2 \cdot 4 = \underline{8}, \\
\mathrm{V}[N] &= \mathrm{V}[\mathrm{E}[N|K]] + \mathrm{E}[\mathrm{V}[N|K]] = \mathrm{V}[2 \cdot K] + \mathrm{E}[K \cdot 1] = 4 \cdot 4 + 4 = \underline{20}.
\end{aligned}$$

2. **Let $X_i \sim \mathrm{Exp}(\lambda_i),\ i = 1, 2, 3$, be independent exponentially distributed random variables. Find**

   a) $\mathrm{P}\{X_1 < X_2 < X_3\}$

   b) $\mathrm{P}\{X_1 < X_2 | \max(X_1, X_2, X_3) = X_3\}$

   Solution:

   a) Directly by integration we obtain

$$\begin{aligned}
\mathrm{P}\{X_1 < X_2 < X_3\} &= \int_0^\infty \mathrm{P}\{X_1 < t\} \cdot \mathrm{P}\{X_3 > t\} \cdot f_2(t)\, dt \\
&= \int_0^\infty (1 - e^{-\lambda_1 t}) e^{-\lambda_3 t} \lambda_2 e^{-\lambda_2 t}\, dt \\
&= \lambda_2 \int_0^\infty e^{-(\lambda_2+\lambda_3)t}\, dt - \lambda_2 \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_3)t}\, dt \\
&= \lambda_2 \left(\frac{1}{\lambda_2+\lambda_3}\right) - \lambda_2 \left(\frac{1}{\lambda_1+\lambda_2+\lambda_3}\right) \\
&= \frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2+\lambda_3)(\lambda_2+\lambda_3)}.
\end{aligned}$$

   b) Generally $\mathrm{P}\{A|B\} = \dfrac{\mathrm{P}\{A \cap B\}}{\mathrm{P}\{B\}}$, which can be applied here.

   Firstly,

$$\begin{aligned}
\mathrm{P}\{\max(X_1, X_2, X_3) = X_3\} &= \mathrm{P}\{X_1 < X_2 < X_3\} + \mathrm{P}\{X_2 < X_1 < X_3\} \\
&= \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2+\lambda_3}\left(\frac{1}{\lambda_1+\lambda_3} + \frac{1}{\lambda_2+\lambda_3}\right) \\
&= \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2+\lambda_3} \cdot \frac{\lambda_1+\lambda_2+2\lambda_3}{(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)} \\
&= \mathrm{P}\{X_1 < X_2 < X_3\} \frac{\lambda_1+\lambda_2+2\lambda_3}{\lambda_1+\lambda_3}.
\end{aligned}$$

   Hence,

$$\begin{aligned}
\mathrm{P}\{X_1 < X_2 | \max(X_1, X_2, X_3) = X_3\} &= \frac{\mathrm{P}\{X_1 < X_2 < X_3\}}{\mathrm{P}\{\max(X_1, X_2, X_3) = X_3\}} \\
&= \frac{\lambda_1+\lambda_3}{\lambda_1+\lambda_2+2\lambda_3}.
\end{aligned}$$

3. **A Markov chain with states $1, \ldots, 4$ has the following transition probability matrix:**

$$\mathbf{P} = \begin{pmatrix} 1-p & 0 & p & 0 \\ q & 0 & q & 0 \\ 0 & 0 & 1-p & p \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

   a) What must the value of q be?

   b) Draw the state transition diagram of the system and classify the states.

   c) What is the probability that the system is in state 4 at time 4 assuming it is in state 2 at time 2?

Solution:

a) The sum of the transition probabilities away from state 2 must be equal to 1,

$$q + q = 1 \qquad \Rightarrow \qquad q = \frac{1}{2}.$$

b) From fig. 2 it can be seen that state 2 is transient and irreducible. The classification of the other states depends on the value of p:

when	**classification**
$p = 0$	states 1 and 3 are absorbing and state 4 transient
$0 < p < 1$	states $1, 3, 4$ form a chain and are positively recurrent (but not periodic). The states of chain also form a closed set.
$p = 1$	states $1, 3, 4$ are positively recurrent and periodic with period of 3.

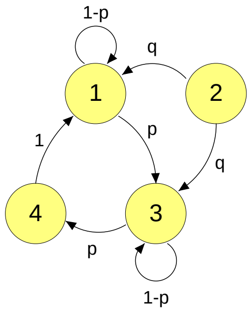


Figure 2: State diagram.

c) First we notice that the initial state is impossible as there it is impossible to be in state 2 at time 2. Assuming that this is still the case, the system is in state 1 on 3^{rd} step with probability of $\frac{1}{2}$, from where the system cannot reach state 4 with one step. Similarly, with probability of $\frac{1}{2}$ the system goes to state 3, from where it moves to state 4 with probability of p. Thus, the asked probability is $\underline{\frac{1}{2}p}$.

4. **Define the state of the system at the n^{th} trial of an infinite sequence of Bernoulli(p) trials to be the number of consecutive succesful trials preceeding and the current trial, i.e. the state is the distance to previous unsuccesful trial. If the n^{th} trial is unsuccesful, then $X_n = 0$; if it succeeds but the previous one was unsuccesful, then $X_n = 1$, etc. a) What is the state space of the system? b) Argue that X_n forms a Markov chain. c) Write down the transition probability matrix of the Markov chain (give its structure).**

Solution:

(a) X_n can be any integer from 0 to ∞

(b) The experiments are independent:

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability of } p, \\ 0 & \text{with probability of } 1-p, \end{cases}$$

that is the state of the system at time $n+1$ depends only on the state at time n. That is the Markov property and hence the system is a Markov chain.

(c)

$$P = \begin{pmatrix} 1-p & p & 0 & 0 & \cdots \\ 1-p & 0 & p & 0 & \cdots \\ 1-p & 0 & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

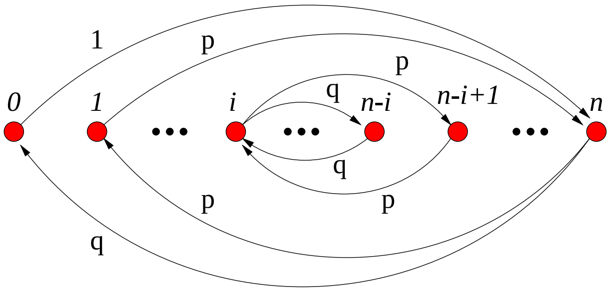
5. **An individual possesses n umbrellas which he employs in going from his home to office and vice versa. If it is raining he will take an umbrella with him, provided there is one to be taken where he is. If it is not raining, then he never takes an umbrella, whatever direction he is going. Assume that, independent of the past and each other, it rains on his way to office or home with probability p. Define the state of the system be i, if where he is there are i umbrellas. Show that**

the equilibrium probabilities of this Markov chain are $\pi_0 = q/(q+n)$ and $\pi_i = 1/(q+n)$, $i = 1, \dots, n$, ' where $q = 1 - p$. Assuming $n = 2$, what value of p maximizes the fraction of time he gets wet?

Solution:

When the system is in state i, there are $n - i$ umbrellas in the other place. Thus, the system will move to state $n - i$ or $n - i + 1$ depending on whether the person takes an umbrella with him or not. This leads to the following transition probability matrix:

$$P = \begin{pmatrix} 0 & 0 & 0 & & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & q & p \\ 0 & 0 & 0 & & q & p & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & q & p & & 0 & 0 & 0 \\ q & p & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$



Substituting the given steady state probabilities gives,

$$\sum_j \pi_j p_{ji} = \begin{cases} \frac{1}{q+n} \cdot q & \text{when } i = 0, \\ \frac{1}{q+n} \cdot q + \frac{1}{q+n} \cdot p = \frac{1}{q+n} & \text{when } 0 < i < n, \\ \frac{q}{q+n} \cdot 1 + \frac{1}{q+n} \cdot p = \frac{1}{q+n} & \text{when } i = n. \end{cases}$$

and thus they satisfy the global balance conditions $\pi_i = \sum_j \pi_j p_{ji}$ and also $\sum_i \pi_i = 1$, and hence are the probabilities in the steady state.

<u>Case $n = 2$</u>:

The proportion of trips he gets wet in the long run is $\pi_0 \cdot p = \frac{q-q^2}{q+2} =: w(q)$. The maximum can be obtained by derivation,

$$w'(q) = \frac{(1-2q)(q+2) - (q-q^2)}{(q+2)^2} = \frac{2 - 4q - q^2}{(q+2)^2}, \qquad 0 \le q \le 1$$

$$w'(q) = 0 \qquad \Leftrightarrow \qquad 2 - 4q - q^2 = 0 \qquad \Rightarrow \qquad q^* = -2 \pm \sqrt{6},$$

thus $p^* = 3 - \sqrt{6} \approx 0.55$ maximizes the probability that person gets wet.

EX 3: Markov, birth death and Poisson processes

1. **a) Consider a Markov chain with n states, states $n_1+1,\ldots,n$ being absorbing states. Argue that the state transition matrix P of the chain is of form**

$$\mathbf{P}=\left(\begin{array}{c|c}\mathbf{A} & \mathbf{B}\\ \hline \mathbf{0} & \mathbf{I}\end{array}\right) \quad \text{and show that} \quad \mathbf{P}^k=\left(\begin{array}{c|c}\mathbf{A}^k & (\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\ldots\mathbf{A}^{k-1})\mathbf{B}\\ \hline \mathbf{0} & \mathbf{I}\end{array}\right),$$

where A on $n_1\times n_1$-matrix, B is $n_1\times n_2$-matrix and I is a unit matrix of proper size.

b) Let $(\pi_k,\ \tilde{\pi}_k)$ denote the state probability vector at the kth step, where π_k corresponds to states $1,\ldots,n_1$ and $\tilde{\pi}_k$ corresponds to states $n_1+1,\ldots,n$. Determine $\tilde{\pi}_k$ and show that $\lim_{k\to\infty}\tilde{\pi}_k=\tilde{\pi}_0+\pi_0\,(\mathrm{I}-\mathrm{A})^{-1}\,\mathrm{B}$.

c) Two persons have a duel, where each side can make one shot at his turn. This is repeated until either duellist hits. The person having the first turn hits with a probability of p_1 and the other person with a probability of p_2. Form a Markov chain corresponding to the duel and determine the winning probabilities of each duellist.

Solution:

a) States $n_1+1,\ldots,n$ were absorbing, i.e. the system remains in such a state forever if it ever enters it, from which the **0**- and **I**-blocks in the partitioned matrix follow.
The latter proposition clearly holds when $k=1$. Assuming that it holds for $k-1$ gives

$$\mathbf{P}^k=\mathbf{P}\cdot\mathbf{P}^{k-1}=\begin{pmatrix}\mathbf{A}^k & \mathbf{A}(\mathbf{I}+\mathbf{A}+\ldots+\mathbf{A}^{k-2})\mathbf{B}+\mathbf{B}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}=\begin{pmatrix}\mathbf{A}^k & (\mathbf{I}+\mathbf{A}+\ldots+\mathbf{A}^{k-1})\mathbf{B}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}.$$

b) Generally the state probability distribution at the kth step is $\boldsymbol{\pi}_0\mathbf{P}^k$. Applying this gives

$$\begin{aligned}(\boldsymbol{\pi}_0\quad \tilde{\boldsymbol{\pi}}_0)\,\mathbf{P}^k &= (\boldsymbol{\pi}_0\quad \tilde{\boldsymbol{\pi}}_0)\begin{pmatrix}\mathbf{A}^k & (\mathbf{I}+\mathbf{A}+\ldots+\mathbf{A}^{k-1})\mathbf{B}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}\\ &= \left(\boldsymbol{\pi}_0\mathbf{A}^k\quad \boldsymbol{\pi}_0(\mathbf{I}+\mathbf{A}+\ldots+\mathbf{A}^{k-1})\mathbf{B}+\tilde{\boldsymbol{\pi}}_0\right),\end{aligned}$$

and hence

$$\tilde{\boldsymbol{\pi}}_k=\boldsymbol{\pi}_0(\mathbf{I}+\mathbf{A}+\ldots+\mathbf{A}^{k-1})\mathbf{B}+\tilde{\boldsymbol{\pi}}_0$$

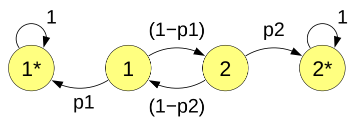
and

$$\lim_{k\to\infty}\tilde{\boldsymbol{\pi}}_k=\boldsymbol{\pi}_0(\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\ldots)\mathbf{B}+\tilde{\boldsymbol{\pi}}_0=\boxed{\tilde{\boldsymbol{\pi}}_0+\boldsymbol{\pi}_0(\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}.}$$

c) A Markov chain corresponding to the duel is illustrated in the figure. The problem can be solved by deduction. Let π_{1^*} denote the probability that the system ends in state 1^*, and similarly, let π_{2^*} denote the respective probability for state 2^*. Due to the lack of memory property of Markov chains one can write

$$\pi_{1^*}=p_1+(1-p_1)\cdot(1-p_2)\cdot\pi_{1^*},$$

from which it follows that



$$\pi_{1^*}=\frac{p_1}{p_1+p_2-p_1p_2} \quad \text{and} \quad \pi_{2^*}=\frac{p_2-p_1p_2}{p_1+p_2-p_1p_2}.$$

Alternatively one can apply the result of b). The state transition matrix is (states in order of $1,\ 2,\ 1^*,\ 2^*$)

$$\mathbf{P}=\begin{pmatrix}0 & 1-p_1 & p_1 & 0\\ 1-p_2 & 0 & 0 & p_2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \quad\Rightarrow\quad \mathbf{A}=\begin{pmatrix}0 & 1-p_1\\ 1-p_2 & 0\end{pmatrix} \text{ ja } \mathbf{B}=\begin{pmatrix}p_1 & 0\\ 0 & p_2\end{pmatrix}.$$

Thus,

$$(\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{p_1+p_2-p_1p_2}\begin{pmatrix} 1 & 1-p_1 \\ 1-p_2 & 1 \end{pmatrix},$$

and as the initial state distribution is $\boldsymbol{\pi}_0 = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $\tilde{\boldsymbol{\pi}}_0 = \begin{pmatrix} 0 & 0 \end{pmatrix}$, one obtains that the probability distribution of the absorbing states in the limit $k \to \infty$ is

$$\lim_{k\to\infty} \tilde{\boldsymbol{\pi}}_k = \frac{\begin{pmatrix} 1 & 1-p1 \end{pmatrix}}{p_1+p_2-p_1p_2} \cdot \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} = \frac{1}{p_1+p_2-p_1p_2} \cdot \begin{pmatrix} p_1 & (1-p1)p_2 \end{pmatrix}.$$

2. **Three persons A, B and C have a truel (cf. duel), where each participant can make a single shot at his turn. A is a good shooter and hits always, B hits with a probability of 2/3 and C with a probability of 1/3. The worst shooter C can start, and after him it is B's turn if he is still alive, and then the turn goes to A if he is still alive, etc. This is repeated until only one person is alive. It is easy to see that it is advantageous for B to always aim at A first, and similarly, for A to aim at B first, if the particular opponent is still alive.**

   a) Determine the probability that C wins assuming that i) he aims at A first, and ii) he aims at C first. Hint: Reduce the problem, with appropriate considerations, to cases where you can apply the result of the case c) of the previous problem.

   b) Even a better alternative for C is to shoot in the air at the first round. Prove it.

Solution:

Let $\mathrm{P}\{C|CA\}$ denote the probability that C wins when only C and A are alive and it is C's turn to shoot next, and similarly for other combinations. Applying the result of c) of the previous problem yields

$$\begin{aligned}
\mathrm{P}\{C|BC\} &= 1 - \frac{2/3}{2/3+1/3-2/9} = \frac{1}{7}, && (p_1 = 2/3,\ p_2 = 1/3) \\
\mathrm{P}\{C|CB\} &= \frac{1/3}{1/3+2/3-2/9} = \frac{3}{7}, && (p_1 = 1/3,\ p_2 = 2/3) \\
\mathrm{P}\{C|CA\} &= 1/3, && (p_1 = 1/3,\ p_2 = 1) \\
\mathrm{P}\{C|AC\} &= 0. && (p_1 = 1,\ p_2 = 1/3)
\end{aligned}$$

Consider next that C misses his first shot. In other words, let $\mathrm{P}\{C|BAC\}$ denote the probability that C wins when all three persons are alive and the next shooter is B. Clearly,

$$\mathrm{P}\{C|BAC\} = \frac{2}{3}\cdot \mathrm{P}\{C|CB\} + \frac{1}{3}\cdot \mathrm{P}\{C|CA\} = \frac{2}{3}\cdot\frac{3}{7} + \frac{1}{3}\cdot\frac{1}{3} = \frac{25}{63} \approx \underline{0.397}.$$

a) i) C's strategy is to aim first at A:

$$\mathrm{P}\{\text{C wins}\} = \frac{1}{3}\cdot \mathrm{P}\{C|BC\} + \frac{2}{3}\cdot \mathrm{P}\{C|BCA\} = \frac{1}{21} + \frac{50}{189} = \frac{59}{189} \approx \underline{0.312}.$$

ii) C's strategy is to aim first at B:

$$\mathrm{P}\{\text{C wins}\} = \frac{1}{3}\cdot \mathrm{P}\{C|AC\} + \frac{2}{3}\cdot \mathrm{P}\{C|BCA\} = \frac{50}{189} =\approx \underline{0.265}.$$

b) The strategy that C shoots in the air at the first round and the shooting turn moves to B was already considered earlier, i.e.

$$\mathrm{P}\{C|BAC\} = \frac{25}{63} \approx \underline{0.397},$$

which is clearly the most advantageous option for C. In fact, it turns out that by first shooting in the air C ensures that he has the highest chance to win the truel! In this case the resulting winning probabilities for A, B and C are $(1/63)\cdot\begin{pmatrix} 14 & 24 & 25 \end{pmatrix}$.

Intuitively: By shooting in the air first C lets B and A first to fight against each other and then he will have the shooting turn against the survivor. If he instead manages to shoot either A or B the survivor of these would have the shooting turn against him.

3. **A continuous time Markov process with states $i = 1, \ldots, 3$ has the following transition rate matrix:**

$$\mathbf{Q} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}.$$

Determine the equilibrium probabilities of the states of this process.

Solution:

For global equilibrium, $\boldsymbol{\pi}$, it holds that $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$. Let distribution be $\boldsymbol{\pi} = (a, b, c)$, when one gets,

$$\begin{aligned} -2a + b + c &= 0, \\ a - b + c &= 0, \\ a - 2c &= 0. \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} a &= 2c, \\ b &= 3c. \end{aligned}$$

Normalization gives,

$$a + b + c = 6c = 1 \qquad \Rightarrow \qquad c = 1/6.$$

Thus the steady state distribution $\boldsymbol{\pi}$ is

$$\boldsymbol{\pi} = \begin{pmatrix} 1/3 & 1/2 & 1/6 \end{pmatrix}.$$

4. **Determine the probability distribution in equilibrium for birth-death processes (state space $i = 0, 1, 2, \ldots$), which transition intensities are a) $\lambda_i = \lambda$, $\mu_i = i\mu$, b) $\lambda_i = \lambda/(i+1)$, $\mu_i = \mu$, where λ and μ are constants.**

Solution:

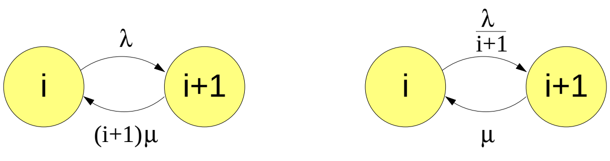


Figure 3: Transition intensities between the states i and $i+1$.

(a) From fig. it can be seen that

$$(i+1)\mu\pi_{i+1} = \lambda\pi_i \qquad \Rightarrow \qquad \pi_{i+1} = \frac{1}{i+1}\frac{\lambda}{\mu}\pi_i = \frac{a}{i+1}\pi_i.$$

That is

$$\begin{aligned} \pi_1 &= a\pi_0, \\ \pi_2 &= \frac{a}{2}\pi_1 = \frac{a^2}{2!}\pi_0, \\ &\vdots \\ \pi_i &= \frac{a^i}{i!}\pi_0. \end{aligned}$$

Furthermore, $\sum_i \pi_i = 1$, i.e.

$$\sum_i \pi_i = \pi_0 \sum_i \frac{a^i}{i!} = \pi_0 e^a \qquad \Rightarrow \qquad \pi_0 = e^{-a}.$$

So

$$\pi_i = \frac{a^i}{i!}e^{-a}.$$

(that is a Poisson-distribution with parameter $a = \lambda/\mu$)

(b) In this case we obtain

$$\frac{\lambda}{i+1}\pi_i = \mu\pi_{i+1} \qquad \Leftrightarrow \qquad \pi_{i+1} = \frac{a}{i+1}\pi_i$$

that is the probability distribution in equilibrium is the same as in (a).

5. **In a game audio signals arrive in the interval $(0, T)$ according to a Poisson process with rate λ, where $T > 1/\lambda$. The player wins only if there will be at least one audio signal in that interval and he pushes a button (only one push allowed) upon the last of the signals. The player uses the following strategy: he pushes the button upon the arrival of the first (if any) signal after a fixed time $s \leq T$.**

a) What is the probability that the player wins?

b) What value of s maximizes the probability of winning, and what is the probability in this case?

Solution:

Player bets on that during the time (s, T) there is exactly one arrival (what happened during $(0, s)$ has no effect here). Let $\tau = T - s$, when the number of arrivals obeys Poisson distribution with parameter $a = \lambda\tau$. Here, $0 \leq \tau \leq T$, i.e. $0 \leq a \leq \lambda T$ where $\lambda T > 1$.

a) $p_v = \mathrm{P}\{N(T) - N(s) = 1\} = \frac{a^1}{1!}e^{-a} = ae^{-a}$, where $a = \lambda(T - s)$.

b) Maximum can be found by taking the first derivate in relative to a:

$$\frac{d}{da}p_v(a) = e^{-a} - ae^{-a} = (1-a)e^{-a}.$$

The derivate has clearly exactly one root, $a = 1$, which is also the maximum of the function (first strictly increasing and then stricly decreasing function).

$$a = 1 \qquad \Rightarrow \qquad \lambda(T-s) = 1 \qquad \Rightarrow \qquad s = T - 1/\lambda,$$

which also lies inside the allowed interval. The maximum probability of winning is hence $\underline{1/e}$.

EX 4: Little's result, Erlang's Loss System

1. **Consider a particular state, i, of a continuous time Markov process as a black box. What is a) the rate of arrivals to this state, b) mean time spent in this state (the life time of the state), c) the average number of systems in this state (note, the system either is or is not in this state). Apply Little's result. Which familiar relation is expressed by the result in this case?**

   Solution:

   a) $\lambda_i = \sum_{j \neq i} \pi_j q_{ji}$

   b) $\overline{W}_i = 1/q_i = \left(\sum_{j \neq i} q_{ij} \right)^{-1}$

   c) There is either "one customer" or no customers: $\overline{N}_i = \pi_i \cdot 1 = \pi_i$.

   Little:

$$\begin{aligned} \overline{N}_i &= \lambda_i \cdot \overline{W}_i \\ \pi_i &= \sum_{j \neq i} \pi_j q_{ji} \cdot 1/q_i \qquad \Rightarrow \qquad \sum_{j \neq i} \pi_i q_{ij} = \sum_{j \neq i} \pi_j q_{ji}. \end{aligned}$$

   The obtained relation is the *global balance equation* for state i, i.e. the total mean flow into the state is equal to the total mean flow out of the state.

2. **The mean time T a car spends in a traffic system is proportional to the number of cars in the system N,**

$$T = \alpha + \beta N^2,$$

   where $\alpha > 0$ and $\beta > 0$ are some constants.

   a) What is the highest possible arrival rate λ^* the system can sustain?

   b) Assuming that the arrival rate is less than λ^*, what is the mean sojourn time T in the system? Is the answer unambiguous? Does the assumption $T = \alpha + \beta N^2$ make sense?

   Solution:

   a) Little's result: $N = \lambda T$.
   Substituting $N^2 = \lambda^2 T^2$ to the original equation yields

$$\lambda^2 \beta T^2 - T + \alpha = 0 \quad \Rightarrow \quad T = \frac{1 \pm \sqrt{1 - 4\alpha\beta\lambda^2}}{2\lambda^2\beta}. \tag{4.1}$$

   The system is stable when T is a finite real number. Thus, the discriminant must be non-negative,

$$1 - 4\alpha\beta\lambda^2 \geq 0 \qquad \Rightarrow \qquad \lambda \leq \frac{1}{2\sqrt{\alpha\beta}}.$$

   Consequently, the maximum possible arrival rate the system can handle is $\underline{\lambda^* = \frac{1}{2\sqrt{\alpha\beta}}}$.

   b) Equation (4.1) has clearly two (stable) solutions (roots). One of them corresponds to a situation where the traffic flows smoothly and the other one to a situation where the traffic has jammed(?).

3. **A modem pool has five modems. 10% of the arriving calls are blocked because all the modems are reserved. What is the traffic intensity (in erlangs) of the offered traffic (assumed to be Poissonian; redial attempts are not considered). How many modems would be needed in order to lower the blocking probability to 1 %?**

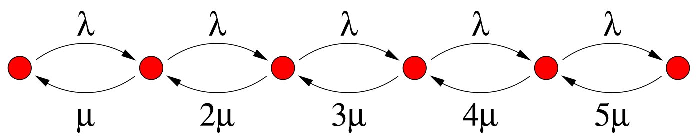


Figure 4: The state space of the system.

Solution:

In equilibrium the probability of state j is

$$\pi_j = \frac{\frac{a^j}{j!}}{1 + \frac{a}{1!} + \frac{a^2}{2!} + \ldots + \frac{a^5}{5!}},$$

where $a = \lambda/\mu$. The time blocking is probability of state $s = 5$,

$$E(s,a) = \frac{\frac{a^5}{5!}}{1 + \frac{a}{1!} + \frac{a^2}{2!} + \ldots + \frac{a^5}{5!}}.$$

If the intensity of arrivals is λ, then the average number of blocked calls is λ times the proportion of the time spent in state 5. Hence $E(5,a) = 10\%$. The approximate value for the offered load a can be obtained for example with Mathematica, or by looking from figure: $\underline{a \approx 2.88}$.

The number of modems required in order to drop the blocking probability to 1% can be obtained by iterating with Mathematica:

```
In[25]:= (a^6/6!)/Sum[ a^i/i!, {i,0,6}]
Out[25]= 0.0458179
In[26]:= (a^7/7!)/Sum[ a^i/i!, {i,0,7}]
Out[26]= 0.0185088
In[27]:= (a^8/8!)/Sum[ a^i/i!, {i,0,8}]
Out[27]= 0.00662153
```

Hence, 8 modems are required.

4. **Ordered search in an n server Erlang system: Assume that the servers are labeled sequentially with numbers $1,\ldots,n$ nd the offered load is a erl. Each arriving customers goes to the free server with the lowest number. What proportion of the time is the server i in use? Hint: Note, that all the servers $1,\ldots,i$ are in use with probability of $E(i,a)$, and deduce from this the average arrival rate to servers $i+1,\ldots,n$, and finally the arrival rate to each server.**

Solution:

Consider the i^{th} server. Because the customers do no switch servers in the middle of service, the group of i first servers acts like normal $M/M/i/i$ system: the arriving customer enters one of the i first servers or is "blocked", when it overflows to servers $i+1,\ldots,n$. Thus, the probability that i first servers are busy is $E(i,a)$.

It follows that the arrival rate of customers to servers $i+1,\ldots,n$ is $\lambda \cdot E(i,a)$. Similarly, the offered traffic to servers $i, i+1,\ldots,n$ is $\lambda \cdot E(i-1,a)$. The difference in offered traffic corresponds to traffic handled by server i, i.e. the rate of customers entering to i^{th} server, λ_i, is

$$\lambda_i = \lambda \left[E(i-1,a) - E(i,a)\right],$$

and the load of the i^{th} server is

$$a_i = \lambda_i/\mu = a\left[E(i-1,a) - E(i,a)\right],$$

which is also the proportion of the time the i^{th} server is in use.

5. **a)** **Consider Erlang's loss system with n servers. The steady state probabilities can be obtained recursively as well by starting from the state n and proceeding towards the origin as vice versa. Let $p_i^{(n)}$ denote the unnormalized state probabilities, $i = 0, \ldots, n$, where we have set $p_n^{(n)}$=1. Let $C(n) = \sum_{i=0}^{n} p_i^{(n)}$ denote the corresponding normalization constant. Show that with this notation the blocking probability is given by $1/C(n)$.**

   b) **Consider next a system with $n+1$ states. Let $p_i^{(n+1)}$, $i = 0, \ldots, n$, denote the respective unnormalized steady state probabilities with $p_{n+1}^{(n+1)}$=1, and let $C(n+1)$ denote the corresponding normalisation constant. In view of the way how the steady state probabilities are determined starting from the state $n+1$, deduce that recursive equation $C(n+1) = 1 + \frac{n+1}{a}C(n)$ holds (which is the recursion formula for Erlang's loss function).**

Solution:

a) Generally it holds for the unnormalized state probabilities $i-1$ and i of Erlang's system that

$$\lambda \cdot p_{i-1}^{(n)} = i\mu \cdot p_i^{(n)} \quad \Rightarrow \quad p_{i-1}^{(n)} = \frac{i}{a} \cdot p_i^{(n)}.$$

In this problem we have chosen $p_n^{(n)} = 1$, which gives

$$p_{i-1}^{(n)} = \left(\frac{i}{a} \cdot \frac{i+1}{a} \cdot \ldots \cdot \frac{n}{a}\right) \cdot p_n^{(n)} = \frac{n!}{(i-1)! \cdot a^{n-i+1}} \cdot 1, \qquad \text{i.e.} \qquad p_i^{(n)} = \frac{n!}{a^n} \cdot \frac{a^i}{i!}.$$

Thus,

$$C(n) = \sum_{i=0}^{n} p_i^{(n)} = \frac{n!}{a^n} \sum_{i=0}^{n} \frac{a^i}{i!},$$

and

$$1/C(n) = \frac{a^n/n!}{\sum_{i=0}^{n} a^i/i!} = \mathrm{Erl}(n, a).$$

b) In recursion, the unnormalized probability of the previous state $i-1$ is obtained by multiplying the unnormalized probability of state i by i/a. Hence, when considering a system with $n+1$ states the unnormalized state probabilities of first n states are the same with some constant factor q (see Fig.):

$$p_i^{(n+1)} = q \cdot p_i^{(n)}, \qquad \text{when } i = 0, \ldots, n.$$

On the other hand, $p_n^{(n)} = 1$ and

$$p_n^{(n+1)} = \frac{n+1}{a} \cdot p_n^{(n)} = \frac{n+1}{a},$$

and hence $q = \frac{n+1}{a}$ and normalization constant $C(n+1)$ for a system with $n+1$ states is given by

$$C(n+1) = q \cdot C(n) + 1 = 1 + \frac{n+1}{a} \cdot C(n).$$

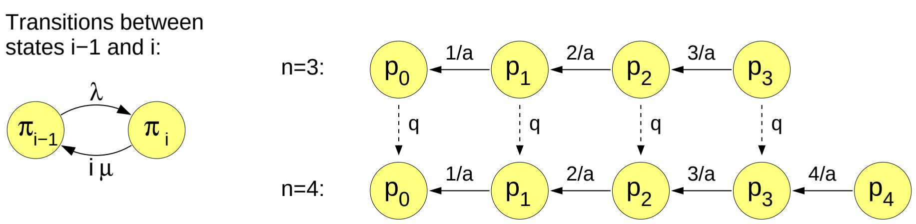


Figure 5: The probability flow between states i and $i-1$ (on left), and recursion for $n = 3\,4$ (on right).

EX 5: Engset's system, M/M/1 and M/M/m

1. **Prove that in Engset's system with k sources and n servers (the $M/M/n/n/k$ system), the probability that arriving customer sees the system in state i, denoted with $\pi_i^*[k]$, is the same as $\pi_i[k-1]$, i.e. the probability of state i in system with $k-1$ customers. Hint: Previously it has been proven that $\pi_i^*[k] = \lambda_i \pi_i[k] / \sum_{j=0}^n \lambda_j \pi_j[k]$.**

   Solution:

   Ratkaisu:

   Let,

$$A_i[k] = \binom{k}{i} p^i (1-p)^{k-i}$$

   when it holds that

$$\begin{aligned} A_i[k] &= \frac{k!}{i!(k-i)!} p^i (1-p)^{k-i} = \frac{k(1-p)}{k-i} \frac{(k-1)!}{i!(k-1-i)!} p^i p^{k-1-i} \\ &= \frac{k(1-p)}{k-i} A_i[k-1]. \end{aligned}$$

   Hence, steady state probabilities $\pi_i[k]$ can be written as $\pi_i[k] = \frac{A_i[k]}{\sum_{k=0}^n A_k[k]}$. Note that the denumerator is independent of i. Next $\pi_i^*[k]$ is determined. Here $\lambda_j = (k-j)\gamma$, so

$$\begin{aligned} \pi_i^*[k] &= \frac{\lambda_i \pi_i[k]}{\sum_{j=0}^n \lambda_j \pi_j[k]} = \left(\frac{\lambda_i A_j[k]}{\sum_{k=0}^n A_k[k]} \right) / \left(\sum_{j=0}^n \frac{\lambda_j A_j[k]}{\sum_{k=0}^n A_k[k]} \right) \\ &= \frac{\lambda_i A_i[k]}{\sum_{j=0}^n \lambda_j A_j[k]} = \frac{(k-i)\gamma \frac{k(1-p)}{k-i} \cdot A_i[k-1]}{\sum_{j=0}^n (k-j)\gamma \frac{k(1-p)}{k-j} \cdot A_j[k-1]} \\ &= \frac{\gamma k(1-p) \cdot A_i[k-1]}{\gamma k(1-p) \sum_{j=0}^n A_j[k-1]} = \frac{A_i[k-1]}{\sum_{j=0}^n A_j[k-1]} \\ &= \pi_i[k-1]. \end{aligned}$$

2. **Consider 2×1- and 4×2-concentrators, where for each input port calls arrive according to independent Poisson-processes with intensities γ. The mean call holding time is denoted by $1/\mu$ and the offered load by $\hat{a} = \gamma/\mu = 0.1$. Compare in these two concentrators the probabilities that a call arriving to a free input port gets blocked because all the output ports are busy.**

   Solution:

   The system corresponds to an Engset's system with n sources and s servers, $0 \le s \le n$. Let $\pi_j[n]$, $j = 0, \ldots, s$, denote the steady state probabilities. Similarly, let $\pi_j^*[n]$, $j = 0, \ldots, s$, denote the state probabilities seen by an arriving customer.

   Time blocking in Engset's system is generally the same as the steady state probability of state s,

$$\pi_s[n] = \frac{\binom{n}{s} \hat{a}^s}{\sum_{k=0}^s \binom{n}{k} \hat{a}^k} \qquad \left(= \frac{\binom{n}{s} p^s (1-p)^{n-s}}{\sum_{k=0}^s \binom{n}{k} p^k (1-p)^{n-k}}, \quad \text{where } p = \frac{\hat{a}}{1+\hat{a}}. \right)$$

Similarly, for call blocking it holds that

$$\pi_s^*[n] = \pi_s[n-1] = \frac{\binom{n-1}{s}\hat{a}^s}{\sum_{k=0}^{s}\binom{n-1}{k}\hat{a}^k}.$$

Substituting n and s to above gives ($p \approx 0.091$)

n	s	time blocking	call blocking
2	1	16.7%	9.1%
4	2	4.1%	2.6%

3. **On average ten customers per hour arrive to a shoe polishing station. The polishing of shoes takes 6 minutes on average. There are two stools, one for the person being served and the other one for a waiting customer. If both stools are occupied, then the arriving customer leaves.**

   a) Draw the state diagram of the system and solve the balance equations, when the arrival process is assumed to be Poissonian and the service times are exponentially distributed. How many customers are served in an hour on average?

   b) What happens if the shoe polisher has an assistant, i.e. when customers on both stools are served at same time and there are no waiting places.

Solution:

Figure 6: State diagrams of M/M/1/2 and M/M/2/2 systems.

The arrival process is Poisson process, so $\lambda = 10$ customers/h $= 1/6$ customers/min. Similarly, the service time is exponentially distributed with $\mu = 1/(6 \text{ min})$. Thus, the offered load a is $a = 1$.

a) $M/M/1/2$ system. From the state diagram it can be deduced that

$$\begin{aligned} \pi_1 &= a\pi_0, \\ \pi_2 &= a\pi_1 = a^2\pi_0 \qquad \Rightarrow \qquad \pi_0 = 1/(1+a+a^2) = 1/3, \end{aligned}$$

and thus,

$$\boldsymbol{\pi} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

The time blocking of the system is $\pi_2 = 1/3$ and from PASTA-property[1] it follows that $2/3$ of the offered traffic gets through, i.e. during one hour $2/3 \cdot 10 \approx 6.67$ customers are served on average.

b) $M/M/2/2$ system (right Fig.). The steady state probabilities in this case are

$$\begin{aligned} \pi_1 &= a\pi_0, \\ \pi_2 &= a/2 \cdot \pi_1 = a^2/2 \cdot \pi_0 \qquad \Rightarrow \qquad \pi_0 = (1/2)/(1+a+a^2/2) = 2/5, \end{aligned}$$

and furthermore,

$$\boldsymbol{\pi} = \begin{pmatrix} 2/5 & 2/5 & 1/5 \end{pmatrix}.$$

Thus, $4/5$ of the arriving customers are accepted and, on average, $4/5 \cdot 10 = 8$ customers are served during one hour.

[1] Poisson arrivals see time averages

4. **Customers arrive to a taxi station according to a Poisson process with intensity of λ. There is room for K taxis in the station and practically for infinite number of customers. Taxis arrive to the station according to a Poisson process with intensity of μ. If the station is full, the taxi drives immediately way. If there is a waiting customer in the station, the taxi picks the customer and otherwise stays to taxi queue waiting. Determine the steady state distribution for both customer and taxi queues. What are the distributions if $\lambda = 1/$min, $\mu = 2/$min and $K = 5$? What is the probability that a customer must wait for a taxi? Hint: Customer and taxi queue cannot exist at the same time. Start writing the state diagram from state 0 when there are no customers or taxis waiting in a queues.**

Solution:

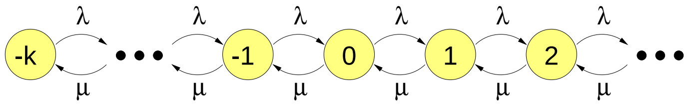


Figure 7: State diagram: $-k, \ldots, -1$ = taxis in queue, $i > 0$ = customers in queue.

As it can be seen from Fig. 7 the system is normal M/M/1 system, where the indexes are relabeled. For steady state probabilities it holds that,

$$\begin{aligned} \mu\pi_{i+1} &= \lambda\pi_i \quad |: \mu \\ \pi_{i+1} &= \rho\pi_i \end{aligned}$$

From which it can be deduced that

$$\pi_i = \rho^{k+i}\pi_{-k}.$$

The sum of probabilities is equal to one,

$$1 = \sum_{i=-k}^{\infty} \rho^{k+i}\pi_{-k} = \pi_{-k}\sum_{i=0}^{\infty}\rho^i = \frac{\pi_{-k}}{1-\rho}.$$

Thus, $\pi_{-k} = 1 - \rho$, ja $\pi_i = \rho^{k+i}(1-\rho)$ kun $-k \le i < \infty$. Substituting the given values $\rho = 1/2$ and $k = 5$ gives $\pi_i = (1/2)^{5+i+1} = \underline{(1/2)^6(1/2)^i}$.

The probability that an arriving customer must wait for a taxi is the sum of states with no taxis in queue, i.e.

$$\sum_{i=0}^{\infty}(1/2)^6(1/2)^i = (1/2)^6\sum_{i=0}^{\infty}(1/2)^i = (1/2)^6\frac{1}{1-1/2} = (1/2)^5 = \underline{1/32}.$$

5. **The output buffers of a router share a common memory area. The arrival streams to the buffers are assumed to be Poissonian and the packet size distribution is exponential. A fixed memory allocation (buffer place) is made for each packet (we assume each packet can be accommodated in this buffer place, in spite of the exponential size distribution), and altogether there are K buffer places available in the memory. What is the probability that all K places are occupied (i.e. the overflow probability of an incoming packet) when there are two output queues sharing the memory, and the loads of the queues are equal ($\rho = 0.7$) and $K = 20$? Compare the result with the case when both buffers have a dedicated memory for 10 packets.**

Solution:

The system consists of two M/M/1-queues, whose state space $(n1, n2)$ is truncated with condition $n_1 + n_2 \le 20$. The probability of state (n_1, n_2) is

$$p(n_1, n_2) = \frac{\rho_1^{n_1}\rho_2^{n_2}}{\Omega} = \frac{\rho^{n_1+n_2}}{\Omega},$$

where Ω is normalization constant. In states where all the memory locations are reserved it holds that $n_1 + n_2 = 20$ and number of such states is 21. Thus, the probability asked is $\underline{21 \cdot 0.7^{20}/\Omega}$.

The normaliztion constant Ω is obtained by taking a sum over all (feasible) states:

$$\begin{aligned}
\Omega &= \sum_{i=0}^{20}\sum_{j=0}^{i} \rho_1^j \rho_2^{i-j} = \sum_{i=0}^{20}(i+1)\rho^i = \sum_{i=0}^{20} \frac{d}{d\rho}\,\rho^{i+1} = \frac{d}{d\rho}\,\sum_{i=0}^{20}\rho^{i+1} \\
&= \frac{d}{d\rho}\,\rho\sum_{i=0}^{20}\rho^i = \frac{d}{d\rho}\,\frac{\rho-\rho^{22}}{1-\rho} = \frac{(1-22\rho^{21})(1-\rho)+\rho-\rho^{22}}{(1-\rho)^2} \\
&= \frac{1-\rho-22\rho^{21}+22\rho^{22}+\rho-\rho^{22}}{(1-\rho)^2} = \frac{1-22\rho^{21}+21\rho^{22}}{(1-\rho)^2} \\
&= \frac{1-22\cdot 0.7^{21}+21\cdot 0.7^{22}}{0.3^2} \approx 11.066.
\end{aligned}$$

Thus, the overflow probability is about 0.0015, i.e. about <u>2 per mil</u>.

In case both buffers had a dedicated memory for 10 packets one obtains the normal truncated M/M/1-queue, for which,

$$p_i = \rho^i p_0.$$

Normalization gives,

$$1 = (1+\rho+\rho^2+\ldots+\rho^{10})p_0 = \frac{1-\rho^{11}}{1-\rho} \qquad \Rightarrow \qquad p_0 = \frac{1-\rho}{1-\rho^{11}}.$$

Thus, the probability of state 10 is

$$p_{10} = \rho^{10}\frac{1-\rho}{1-\rho^{11}} = \frac{\rho^{10}-1+1-\rho^{11}}{1-\rho^{11}} = 1 - \frac{1-\rho^{10}}{1-\rho^{11}} \approx 0.009 \approx \underline{1\%}.$$

EX 6: P-K formula and priority queues

1. **Consider a M/G/1 queue. Prove, by using the Little's theorem, that for any work conserving queueing discipline it holds that,**
$$\mathbf{P}\{N > 0\} = \rho.$$

Solution:

Work conserving queueing discipline means simply that when there is customers in the system then they are served. Let N_s be the number of customers in the server, i.e. $N_s \in \{0, 1\}$.

$$\begin{aligned}
\mathrm{P}\{N > 0\} &= \mathrm{P}\{N_s = 1\} && \text{work conserving queueing discipling}\\
&= \mathrm{E}\left[N_s\right] && \text{definition of expectation}\\
&= \lambda \mathrm{E}\left[S\right] && \text{Little's formula, } S \text{ is the service time}\\
&= \rho.
\end{aligned}$$

2. **Carloads of customers arrive at a single-server station in accordance with a Poisson process with rate 4 per hour. The service times are exponentially distributed with mean 3 min. If each carload contains either 1, 2, or 3 customers with respective probabilities $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$, compute the average customer waiting time in the queue. Hint: The waiting time of the first customer of each group can be obtained from an appropriate $M/G/1$ queue. Consider separately the "internal" waiting time in the group.**

Solution:

Consider first the whole batch as one unit and later the waiting time inside batch can be added to it. For the whole batch one can apply M/G/1-queue model. It was given that

$$\lambda = 4/\mathrm{h} = 1/15\mathrm{min}, \qquad \text{and} \qquad 1/\mu = 3\mathrm{min}.$$

The service time of the batch is,

$$S = X_1 + \ldots + X_N, \qquad \text{where } N \text{ is 1, 2 or 3.}$$

For the number of customers, N, in one group it holds that,

$$\begin{aligned}
\mathrm{E}\left[N\right] &= 1 \cdot (1/4) + 2 \cdot (1/2) + 3 \cdot (1/4) = 2\\
\mathrm{V}\left[N\right] &= \mathrm{E}\left[(N - \mathrm{E}\left[N\right])^2\right] = 1 \cdot (1/4) + 0 \cdot (1/2) + 1 \cdot (1/4) = 1/2.
\end{aligned}$$

from which, by using the conditioning rule (or tower property), one gets for the service time S that,

$$\begin{aligned}
\mathrm{E}\left[S\right] &= \mathrm{E}\left[\mathrm{E}\left[S|N\right]\right] = \mathrm{E}\left[N \cdot 3\mathrm{min}\right] = 6\mathrm{min},\\
\mathrm{V}\left[S\right] &= \mathrm{E}\left[\mathrm{V}\left[S|N\right]\right] + \mathrm{V}\left[\mathrm{E}\left[S|N\right]\right] = \mathrm{E}\left[N \cdot 9\mathrm{min}^2\right] + \mathrm{V}\left[N \cdot 3\mathrm{min}\right]\\
&= 18\mathrm{min}^2 + \frac{9}{2}\mathrm{min}^2 = 45/2\mathrm{min}^2\\
\mathrm{E}\left[S^2\right] &= \mathrm{V}\left[S\right] + \mathrm{E}\left[S\right]^2 = 45/2\mathrm{min}^2 + 36\mathrm{min}^2 = \frac{117}{2}\mathrm{min}^2.
\end{aligned}$$

The average waiting time of batch can be obtained by using the Pollaczek-Khinchin formula,

$$\mathrm{E}\left[W_g\right] = \frac{\lambda \mathrm{E}\left[S^2\right]}{2(1 - \lambda \mathrm{E}\left[S\right])} = \frac{\frac{1}{15\mathrm{min}}\frac{117}{2}\mathrm{min}^2}{2(1 - \frac{1}{15\mathrm{min}}6\mathrm{min})} = \frac{\frac{117}{30}}{\frac{6}{5}}\mathrm{min} = \frac{13}{4}\mathrm{min} = 3\mathrm{min}\ 15\mathrm{s}.$$

In addition the customer sometimes have to wait within his batch. This is easiest to obtain by determining the total waiting time inside one batch and then dividing that by the average size of the batch:

$$\mathrm{E}[W_c] = (1/4 \cdot 0 + 1/2 \cdot 1 + 1/4 \cdot 3) \cdot 3\mathrm{min}/2 = 15/8\mathrm{min} = 1\mathrm{min}\ 52.5\mathrm{s}.$$

(Note. When batch size is 3, the average waiting time is $(0+1+2)\cdot 3\mathrm{min} = 3 \cdot 3\mathrm{min}$.)

By adding the waiting times together one gets the total average waiting time of customer, which is

$$\mathrm{E}[W] = \mathrm{E}[W_g] + \mathrm{E}[W_c] = \underline{5\mathrm{min}\ 7.5\mathrm{s}}.$$

3. **A generalization of Little's result. Consider an arrival-departure system with arrival rate λ, where entering customers are forced to pay money to the system according to some rule.**

   a) Argue that the following identity holds:

   $$\textbf{Averate rate at which the system earns} = \boldsymbol{\lambda} \cdot \textbf{(Average amount a customer pays).}$$

   Show that Little's theorem is a special case of above.

   b) Consider the $M/G/1$ system and the following cost rule: Each customer pays at a rate of y per unit time when its remaining service time is y, whether in queue or in service. Show that the formula in a) can be written as (cf. Pollaczek-Khinchin formula)

   $$\boldsymbol{\overline{W} = \lambda\,(\overline{X}\,\overline{W} + \overline{X^2}/2)},$$

   where W denotes the waiting time of a customer and X denotes the service time.

Solution:

a)

Consider the time interval $(0,t)$. Let

M_t	no. of departing customers during $(0,t)$
A_t	total cumulative income during $(0,t)$
m	mean payment by a customer

$$E[A_t] = E[E[A_t|M_t]] = E[M_t \cdot m] = m\lambda t$$

Hence, the mean income rate is

$$E[A_t]/t = m\lambda.$$

Little's result follows when the payment is chosen to be equal to the sojourn time in the system, $m = \overline{T}$. Then, the charge rate is equal to one and, consequently, the income rate is equal to the number of customers at the system, N.

$$\underbrace{\overline{N}}_{\text{mean income rate}} = \underbrace{\overline{T}}_{\text{mean payment}} \cdot \lambda.$$

b)

M/G/1-queue with charge rate = the remaining service time of customer. The mean payment in this case is

$$m = \overline{X} \cdot \overline{W} + \frac{1}{2}\overline{X^2},$$

where X and W are independent random variables.

Income rate of the system is equal to the sum of the remaining service times of the customers in the system, i.e. the unfinished work of the system, denoted by U, i.e.

$$U = U_1 + U_2 + \ldots + U_N + R,$$

The following holds for the power series:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \text{and} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Let $X \sim$ Uniform$(1, n)$ and $Y = m + X$. Then,

$$\begin{aligned} \mathrm{E}[X] &= \tfrac{n+1}{2} & \mathrm{E}[Y] &= m + \tfrac{n+1}{2} \\ \mathrm{E}\left[X^2\right] &= \tfrac{(n+1)(2n+1)}{6} & \mathrm{E}\left[Y^2\right] &= m^2 + 2m\,\mathrm{E}[X] + \mathrm{E}\left[X^2\right] \\ & & &= m^2 + m(n+1) + \tfrac{(n+1)(2n+1)}{6} \end{aligned}$$

Table 2: The two first moments of discrete uniform distribution.

where R corresponds to the remaining service time of the customer currently being served, and U_i corresponds to service time of customers i. This is the same as the virtual waiting time, i.e. the waiting time of a customer which arrives at the system at a given point of time. The Poisson-arrivals have the lack of memory property from which it follows that the distribution of the actual waiting time W is the same.

W	unfinished work at the queue when customer arrives
U	unfinished work at the queue at an arbitrary point of time

In other words, $U \sim W$ and $\overline{U} = \overline{W}$, and hence, $\overline{W} = m\lambda$. Furthermore, it holds that

$$\overline{W} = \lambda m = \lambda(\overline{X} \cdot \overline{W} + \frac{1}{2}\overline{X^2}),$$

from which we obtain

$$\overline{W}(1 - \lambda\overline{X}) = \frac{\lambda\overline{X^2}}{2} \qquad \Rightarrow \qquad \overline{W} = \frac{\lambda\overline{X^2}}{2(1 - \lambda\overline{X})} = \frac{\lambda\overline{X^2}}{2(1-\rho)},$$

which is P-K formula.

4. **A grocery store has two checkout counters each of which together receives a Poisson stream of customers with rate 1/2 per minute. The number of items each customer carries is uniformly distributed in the range $1, \ldots, 30$. Processing an item at the counter takes 4 s. a) What is the average waiting time of an customer in the checkout queue? b) Write the expression for the mean waiting time in the case that counter 1 servers customers with no more than k items, and counter 2 is dedicated for customer with more than k items. Find the optimal value for k with the aid of a small computer program.**

Solution:

The first moments of uniform distribution are presented in Table 2.

a) P-K formula can be applied: $\mathrm{E}[W] = \frac{\lambda \mathrm{E}\left[S^2\right]}{2(1-\rho)}$.
Counters behave identically so we can concentrate on the first counter. The arrival rate λ to first counter is

$$\lambda = \frac{1}{2}/\text{min} = \frac{1}{120\text{ s}}.$$

Let X be the number of items, i.e. X is uniformly distributed in the range $1, \ldots, 30$. Then the service time S of customer is

$$S = 4\text{ s} \cdot X.$$

It follows

$$\begin{aligned}
\mathrm{E}[S] &= 4\text{ s}\cdot\mathrm{E}[X] = 4\text{ s}\cdot 31/2 = 62\text{ s},\\
\mathrm{E}\left[S^2\right] &= 16\text{ s}^2\cdot\mathrm{E}\left[X^2\right] = 16\text{ s}^2\cdot\sum_{i=1}^{30}\frac{1}{30}i^2 = 16\text{ s}^2\cdot\frac{30\cdot 31\cdot 61}{6\cdot 30} = 15128/3\text{ s}^2 \approx 5043\text{ s}^2.\\
\rho &= \lambda\cdot\mathrm{E}[S] = \frac{1}{120\text{ s}}\cdot 62\text{ s} = \frac{31}{60} =\approx 0.5167.
\end{aligned}$$

Substituting the numerical values to P-K formula gives

$$\mathrm{E}[W] = \frac{\lambda\mathrm{E}\left[S^2\right]}{2(1-\rho)} = \frac{\frac{1}{120\text{ s}}\cdot 15128/3\text{ s}^2}{2(1-31/60)} = \frac{3782\text{ s}}{87} \approx \underline{43.5\text{ s}}.$$

b) Here the total arrival rate $\lambda_{\text{tot}} = 1/\text{min}$ is split to two counters according to the number of items X. Customers who buy most k items are forwarded to counter 1, and the rest to counter 2. Hence, the parameters of counter 1 are

$$\begin{aligned}
p_1 &= \mathrm{P}\{\text{arriving customer is served at counter 1}\} = \frac{k}{30},\\
\lambda_1 &= \frac{k}{30\cdot 60\text{ s}} = \frac{k}{1800\text{ s}},\\
\mathrm{E}[S_1] &= 4\text{ s}\cdot\mathrm{E}[X_1] = 4\text{ s}\cdot(k+1)/2 = 2(k+1)\text{ s},\\
\mathrm{E}\left[S_1^2\right] &= 16\text{ s}^2\cdot\mathrm{E}\left[X_1^2\right] = 16\text{ s}^2\cdot\sum_{i=1}^{k}\frac{1}{k}i^2 = 16\text{ s}^2\cdot\frac{k\cdot(k+1)\cdot(2k+1)}{6\cdot k} = \frac{8(k+1)(2k+1)}{3}\text{ s}^2,\\
\rho_1 &= \lambda_1\cdot\mathrm{E}[S_1] = \frac{k}{1800\text{ s}}\cdot 2(k+1)\text{ s} = \frac{k^2+k}{900},
\end{aligned}$$

Similarly, the parameters of counter 2 are ($k_2 = 30-k$):

$$\begin{aligned}
p_2 &= \mathrm{P}\{\text{arriving customer is served at counter 2}\} = \frac{30-k}{30},\\
\lambda_2 &= \frac{30-k}{30\cdot 60\text{ s}} = \frac{30-k}{1800\text{ s}},\\
\mathrm{E}[S_2] &= 4\text{ s}\cdot\mathrm{E}[X_2] = 4\text{ s}\cdot[k+(k_2+1)/2] = (2k+62)\text{ s},\\
\mathrm{E}\left[S_2^2\right] &= 16\text{ s}^2\cdot\mathrm{E}\left[X_1^2\right] = 16\text{ s}^2\cdot\sum_{i=k+1}^{30}\frac{1}{30-k}i^2 = 16\text{ s}^2\cdot\left[k^2+2k\mathrm{E}[X]+\mathrm{E}\left[X^2\right]\right]\\
\rho_2 &= \lambda_1\cdot\mathrm{E}[S_1] = \frac{30-k}{1800\text{ s}}\cdot(2k+62)\text{ s} = \frac{1860-2k-2k^2}{1800}
\end{aligned}$$

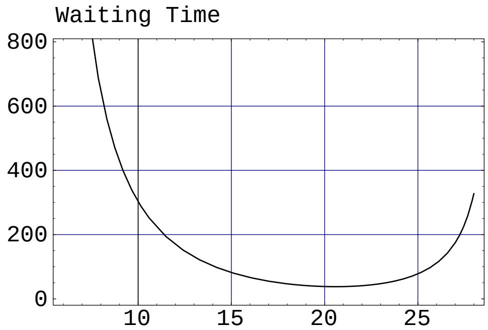


Figure 8: The average waiting time of arriving customer ($\overline{W} = p_1\cdot\overline{W}_1 + p_2\cdot\overline{W}_2$).

P-K formula gives class specific mean waiting times. The optimal k is $k = 21$, which gives an average waiting time $\overline{W} \approx \underline{38.2\text{ s}}$. Below the solution is obtained with Mathematica.

```
'X Uniformly distributed  in 1 ... k'

EX[k_]  := (k + 1)/2
EX2[k_] := (k + 1)*(2*k + 1)/6

'S Uniformly distributed in k1 ...k2'

ES[k1_,  k2_] := 4*(k1 - 1 + EX[k2 - k1 + 1])
ES2[k1_, k2_] := 16*((k1 - 1)^2 + 2*(k1 - 1)*EX[k2 - k1 + 1] + EX2[k2 - k1 + 1])

'P - K formula for waiting time'

W[l_, S_, S2_] := l*S2/(2*(1 - l*S))

'a)'

W[1/120, ES[1, 30], ES2[1, 30]] // N
Out[37] = 43.4713

'b'

Wk[k_] := k/30*W[k/1800, ES[1, k], ES2[1, k]] + (30 - k)/30*
      W[(30 - k)/1800, ES[k + 1, 30], ES2[k + 1, 30]]
Table[Wk[k], {k, 16, 22}] // N
Out[51]= {66.4737, 54.9805, 46.7428, 41.3038, 38.4573, 38.2329, 40.9373}
```

5. **Consider a simplified model for TCP link.**[2] **Assume that TCP packets arrive according to a Poisson process with arrival intensity of $\lambda = 100$ pkt/s to a 2 Mbit/s DSL-modem acting as a router. The packet length distribution and respective service times are the following:**

length	proportion	time / ms
40	0.1	0.16
576	0.3	2.3
1500	0.6	5.9

Determine the mean waiting time of a packet in the queue, when the service discipline is,

a) FIFO

b) the shortest job first (non-preemptive)

Solution:

The arrival rate of packets was$100/\text{s}$, i.e. $0.1/\text{ms}$.

a) Apply P-K mean time formula:

$$\mathrm{E}[S] = 0.1 \cdot 0.16 + 0.3 \cdot 2.3 + 0.6 \cdot 5.9 = 4.246 \text{ ms},$$
$$\mathrm{E}\left[S^2\right] = 0.1 \cdot 0.16^2 + 0.3 \cdot 2.3^2 + 0.6 \cdot 5.9^2 \approx 22.5 \text{ ms}^2.$$

Thus,

$$\rho = \lambda \mathrm{E}[S] = 0.4246,$$

and P-K mean time formula gives,

$$\overline{W} = \frac{\lambda \mathrm{E}\left[S^2\right]}{2(1-\rho)} = \frac{0.1 \cdot 22.5}{2(1-0.4246)} \approx \underline{1.95 \text{ ms}}.$$

b) In this case the packets are classified into three classes according to the length:

length	proportion	λ	$E[S_i]$	$\mathrm{E}\left[S_i^2\right]$
40	0.1	0.01	0.16	0.0256
576	0.3	0.03	2.3	5.29
1500	0.6	0.06	5.9	34.81

[2] Typically there are three peaks in TCP packet length distribution: the first peak at 40 bytes (ACK), the second peak around 552/576 bytes (the smallest possible value for MTU) and the third at 1500 bytes (the largest possible IP packet in ethernet).

The mean remaining service time $\overline{R}$ is,

$$\overline{R} = \frac{1}{2} \sum_k \lambda_k \overline{S_k^2} \approx 1.124 \text{ ms.}$$

So the class-specific waiting times are,

$$\begin{aligned}
\overline{W}_1 &= \frac{\overline{R}}{1-\rho_1} \approx 1.126, \\
\overline{W}_2 &= \frac{\overline{R}}{(1-\rho_1)(1-\rho_1-\rho_2)} \approx 1.211, \\
\overline{W}_3 &= \frac{\overline{R}}{(1-\rho_1-\rho_2)(1-\rho_1-\rho_2-\rho_3)} \approx 2.10.
\end{aligned}$$

From what one gets the average waiting time,

$$\overline{W} = \sum_i p_i \overline{W}_i \approx \underline{1.74 \text{ ms}}.$$

EX 7: M/G/1-queue, time reversibility

1. **The Pollaczek-Khinchin formula for the Laplace transform of the waiting time W is**

$$W^*(s) = \frac{s(1-\rho)}{s - \lambda + \lambda S^*(s)}$$

where $S^*(s)$ is the Laplace transform of the service time S. Apply the transform formula to the $M/D/1$ system, where the service time is constant d. Calculate the expectation and variance of the waiting time. Hint: Determine $S^*(s)$, develop it into power series, take an appropriate number of terms and make the division.

Solution:

M/G/1 queue where the service time is a constant d. Hence, $S^*(s) = e^{-sd}$ and PK formula gives

$$W^*(s) = \frac{s(1-\rho)}{s - \lambda + \lambda e^{-sd}}.$$

Furthermore,

$$\begin{cases} E[W] &= -W^{*\prime}(0), \\ E[W^2] &= W^{*\prime\prime}(0) \end{cases} \quad \Rightarrow \; V[W] = W^{*\prime\prime}(0) - W^{*\prime}(0)^2$$

Developing W^* to a Taylor's serie gives $W^*(s) = W^*(0) + W^{*\prime}(0)s + \frac{1}{2}W^{*\prime\prime}(0)s^2 \ldots$. On the other hand $W^*(s)$ is

$$\begin{aligned} W^*(s) &= \frac{s(1-\rho)}{s - \lambda + \lambda(1 - ds + \frac{1}{2}d^2s^2 - \frac{1}{6}d^3s^3 \ldots)} = \frac{s(1-\rho)}{s(1-\rho) + \frac{1}{2}\lambda d^2 s^2 - \frac{1}{6}\lambda d^3 s^3 \ldots} \\ &= \frac{1}{1 + \frac{\lambda d^2}{2(1-\rho)}s - \frac{\lambda d^3}{6(1-\rho)}s^2 \ldots} = 1 - \frac{\lambda d^2}{2(1-\rho)}s + \left[\left(\frac{\lambda d^2}{2(1-\rho)}\right)^2 + \frac{\lambda d^3}{6(1-\rho)}\right] s^2 \ldots \\ &\approx 1 - \frac{\lambda d^2}{2(1-\rho)}s + \frac{\lambda d^3}{2(1-\rho)}\left[\frac{\rho}{2(1-\rho)} + \frac{1}{3}\right] s^2 = 1 - \frac{\lambda d^2}{2(1-\rho)}s + \frac{\lambda d^3(\rho+2)}{12(1-\rho)^2}s^2. \end{aligned}$$

Comparing the multipliers of the series gives ($\lambda a = \rho$)

$$W^{*\prime}(0) = -\frac{\rho}{2(1-\rho)}d \qquad \text{and} \qquad W^{*\prime\prime}(0) = \frac{\rho(\rho+2)}{6(1-\rho)^2}d^2.$$

Thus,

$$\begin{aligned} \mathrm{E}[W] &= \frac{\rho}{2(1-\rho)}d \\ \mathrm{V}[W] &= \frac{\rho(\rho+2)}{6(1-\rho)^2}d^2 - \left(\frac{\rho}{2(1-\rho)}\right)^2 d^2 = \frac{4\rho + 2\rho^2}{12(1-\rho)^2}d^2 - \frac{3\rho^2}{12(1-\rho)^2}d^2 \\ &= \frac{\rho(4-\rho)}{12(1-\rho)^2}d^2. \end{aligned}$$

2. **Consider a M/G/1-queue. Let S denote the customers service time and $F_S(t)$ its cumulative distribution function. Furthermore, let R denote the customers remaining service time on condition that there is a customer in the server. Show that probability density function of R is**

$$f_R(t) = \frac{1 - F_S(t)}{\mathrm{E}[S]}.$$

Hint: generalize the hitchikers paradox presented in the lecture notes from mean times to pdf.

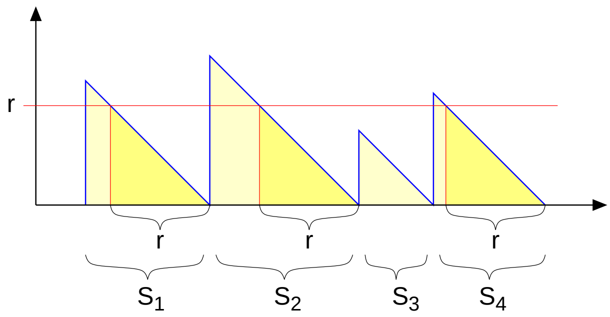


Figure 9: Behaviour of conditioned system.

Solution:

Let S' be the service time of the customer in service at arbitrary point of time on condition that there is a customer in server. The original system is sometimes empty but in the conditioned system there is always a customer in the server and the service time behaves according to figure 9.

From figure one can deduce that it is more probable to “hit” on a longer service time than shorter. Next we try to deduce the probability that at an arbitrary point of time the service time S' of current customer would be in the interval $(x, x+dx)$. This is clearly proportional to[3] a) the length of interval, x, and to b) probability that service time would in of given length, $\mathrm{P}\{S \in (x, x+dx)\}$. Thus,

$$\begin{aligned} & \mathrm{P}\{S' \in (x, x+dx)\} \propto x \cdot \mathrm{P}\{S \in (x, x+dx)\} = x f_S(x) \\ \Longrightarrow \qquad & \mathrm{P}\{S' \in (x, x+dx)\} = \frac{x f_S(x)}{\int y f_S(y)\, dy} = \frac{x}{\mathrm{E}[S]} f_S(x). \end{aligned}$$

Next we determine the tail distribution of R:

$$\begin{aligned} \mathrm{P}\{R > r\} &= \int_0^\infty \mathrm{P}\{R > r | S' = x\} \cdot \mathrm{P}\{S' \in (x, x+dx)\}\, dx \\ &\overset{\text{uniform}}{=} \int_r^\infty \frac{x-r}{x} \cdot \frac{x}{\mathrm{E}[S]} f_S(x)\, dx \\ &= \frac{1}{\mathrm{E}[S]} \left(\int_r^\infty x f_S(x)\, dx - r\mathrm{P}\{S > r\} \right), \end{aligned}$$

and taking a derivate in respect to r gives,

$$\begin{aligned} \frac{d}{dr}\mathrm{P}\{R > r\} &= \frac{1}{\mathrm{E}[S]} \left(-r f_S(r) + r f_S(r) - \mathrm{P}\{S > r\} \right) = -\frac{\mathrm{P}\{S > r\}}{\mathrm{E}[S]} \\ f_R(r) = -\frac{d}{dr}\mathrm{P}\{R > r\} &= \frac{\mathrm{P}\{S > r\}}{\mathrm{E}[S]} = \frac{1 - \mathrm{P}\{S < r\}}{\mathrm{E}[S]} = \boxed{\frac{1 - F_S(r)}{\mathrm{E}[S]}.} \end{aligned}$$

3. **Consider an $M/G/1$ queue. Let the number of new customers arriving during a service time S be V and denote $a_i = \mathrm{P}\{V > i\}$, $i = 0, 1, \ldots$. Show that $\mathrm{E}[V] = \lambda \mathrm{E}[S]$ and further that $\rho = \lambda \mathrm{E}[S] = \sum_{i=0}^\infty a_i$.**

Solution:

Applying the conditioning rule of expectation gives

$$\mathrm{E}[V] = \mathrm{E}[\mathrm{E}[V|S]] = \mathrm{E}[\lambda S] = \lambda \mathrm{E}[S] = \rho.$$

[3]Take a big number N of service times S_i. From these on average $N \cdot \mathrm{P}\{S \in (x, x+dx)\}$ fall in the range $(x, x+dx)$. Thus, the probability to “hit” on service time with given length is $C \cdot x \cdot \mathrm{P}\{S \in (x, x+dx)\}$, where C is a normalization constant.

Similarly,

$$\rho = \mathrm{E}[V] = \sum_{i=0}^{\infty} i \cdot \mathrm{P}\{V = i\} = \sum_{i=1}^{\infty} \mathrm{P}\{V = i\} + \sum_{i=2}^{\infty} \mathrm{P}\{V = i\} + \ldots = \sum_{i=0}^{\infty} a_i.$$

4. **In a finite $M/G/1/4$ queue the arrival rate λ and the distribution of the service time S are such that $a_i = \mathrm{P}\{V > i\} = 1/3^{i+1}$. a) Based on the result of the previous problem, calculate the load ρ of the queue. b) Determine the queue length distribution of the system. Hint: Solve the unnormalized equilibrium probabilities in an infinite system $\pi_i^{(\infty)}$ $(i = 0, \ldots, 3)$ using the recursion on page 29, setting e.g. $\pi_0^{(\infty)} = 1$. Calculate the probabilities of the queue length after a departure in a finite system $\pi_i^{(K)}$ $(i = 0, \ldots, 3)$ by using the formula on page 31, and finally, the queue length probabilities in a finite system p_i $(i = 0, \ldots, 4)$ with the aid of the formulae on page 32.**

Solution:

a) Here $a_i = \mathrm{P}\{V > i\} = \frac{1}{3^{i+1}}$. Hence,

$$\rho = \sum_{i=0}^{\infty} a_i = \sum_{i=0}^{\infty} \frac{1}{3^{i+1}} = \frac{1}{3}\frac{1}{1-\frac{1}{3}} = \frac{1}{2}.$$

b) The point probabilities k_i are

$$k_i = \mathrm{P}\{V = i\} = a_{i-1} - a_i = \frac{2}{3^{i+1}}.$$

The steady state probabilities of M/G/1 queue can be obtained by using the following recursive formula:

$$\pi_i^{(\infty)} = \frac{1}{k_0}\left(a_{i-1}\pi_0^{(\infty)} + \sum_{j=1}^{i-1} a_{i-j}\pi_j^{(\infty)}\right).$$

Substituting a_i and k_i, and choosing $\pi_0^{(\infty)} = 1$, it follows

$$\pi_i^{(\infty)} = \frac{3}{2}\left(\frac{1}{3^i} + \sum_{j=1}^{i-1} \frac{1}{3^{i-j+1}}\pi_j^{(\infty)}\right) = \frac{1}{2 \cdot 3^{i-1}}\left(1 + \sum_{j=1}^{i-1} 3^{j-1}\pi_j^{(\infty)}\right).$$

Applying the recursion gives

$$\begin{aligned}
\pi_0^{(\infty)} &= 1, \\
\pi_1^{(\infty)} &= \frac{1}{2 \cdot 3^0} = \frac{1}{2}, \\
\pi_2^{(\infty)} &= \frac{1}{2 \cdot 3^1}\left(1 + 3^0\pi_1^{(\infty)}\right) = \frac{1}{4} \\
\pi_3^{(\infty)} &= \frac{1}{2 \cdot 3^2}\left(1 + 3^0\pi_1^{(\infty)} + 3^1\pi_2^{(\infty)}\right) = \frac{1}{2 \cdot 9}\left(\frac{4+2+3}{4}\right) = \frac{1}{8}.
\end{aligned}$$

The queue length distribution after a *departing customer* is obtained by normalizing the above probabilities:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{8+4+2+1}{8} = \frac{15}{8}.$$

Hence,

$$\boldsymbol{\pi}^{(K)} = \frac{1}{15}\begin{pmatrix} 8 & 4 & 2 & 1 \end{pmatrix}.$$

The queue length distribution an *arriving customer* sees is obtained from:

$$p_i = \frac{\pi_i^{(K)}}{\rho + \pi_0^{(K)}}, \qquad i = 0, 1, \ldots, K-1 \qquad \text{and} \qquad p_K = 1 - \frac{1}{\rho + \pi_0^{(K)}}.$$

Hence, $\rho + \pi_0^{(K)} = \frac{1}{2} + \frac{8}{15} = \frac{31}{30}$ and substituting the numerical values to equations gives

$$\mathbf{p} = \frac{1}{31}\begin{pmatrix} 16 & 8 & 4 & 2 & 1 \end{pmatrix}.$$

5. **Let $B^*(s)$ denote the Laplace transform of the busy period B in an $M/G/1$ queue. This satisfies so-called Takács' equation,**

$$\boldsymbol{B^*(s) = S^*(s + \lambda - \lambda B^*(s)),}$$

where $S^*(s)$ is the Laplace transform of the service time S.

a) Derive expression for $B^*(s)$ in the case of an $M/M/1$ queue, i.e. when $S^*(s) = \mu/(\mu + s)$.

b) Derive the following expressions for the first two moments of B:

$$\boldsymbol{\overline{B} = \frac{\overline{S}}{1-\rho}, \qquad \overline{B^2} = \frac{\overline{S^2}}{(1-\rho)^3}.}$$

Solution:

a)

A direct substitution gives

$$\begin{aligned}
B^*(s) &= S^*(s + \lambda - \lambda B^*(s)) = \frac{\mu}{\mu + s + \lambda - \lambda B^*(s)} \\
0 &= -\lambda \left(B^*(s)\right)^2 + (\mu + s + \lambda) B^*(s) - \mu \\
B^*(s) &= -\frac{\mu + s + \lambda}{2\lambda} \pm \frac{\sqrt{(\mu + s + \lambda)^2 - 4\lambda\mu}}{2\lambda} \\
&= -\frac{\mu + s + \lambda - \sqrt{(\mu + s + \lambda)^2 - 4\lambda\mu}}{2\lambda},
\end{aligned}$$

because in order to have a stable solution we must have $|B^*(s)| \leq 1$.

b)

$$\overline{B} = -\frac{d}{ds} B^*(s)_{|s=0}$$

the first derivate of $B^*(s)$ is

$$\begin{aligned}
-\frac{d}{ds} B^*(s) &= -\frac{d}{ds}\left(S^*(s + \lambda - \lambda B^*(s))\right) \\
&= -\frac{dS^*\left(s + \lambda - \lambda B^*(s)\right)}{ds} \cdot \left(1 - \lambda \frac{dB^*(s)}{ds}\right)
\end{aligned}$$

When $s = 0$, we have $B^*(0) = 1$ and

$$\begin{aligned}
\overline{B} &= -\frac{dS^* \overbrace{(0 + \lambda - \lambda B^*(0))}^{=0}}{ds} \cdot \left(1 + \lambda \overline{B}\right) \\
&= \overline{S} + \lambda \overline{S}\overline{B} \\
\overline{B} &= \frac{\overline{S}}{1 - \lambda \overline{S}} = \frac{\overline{S}}{1-\rho}.
\end{aligned}$$

The second moment can be obtained from the 2nd derivate,

$$\overline{B^2} = \frac{d^2}{ds^2} B^*(s)_{|s=0}$$

i.e.

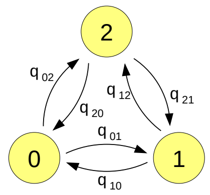
$$\begin{aligned}
\frac{d^2}{ds^2} B^*(s) &= \frac{d^2}{ds^2} S^*\left(s + \lambda - \lambda B^*(s)\right) \left(1 + \lambda \left(-\frac{d}{ds} B^*(s)\right)\right)^2 + \\
&\quad + \frac{d}{ds} S^*\left(s + \lambda - \lambda B^*(s)\right) \left(-\lambda \frac{d^2}{ds^2} B^*(s)\right)
\end{aligned}$$

Furthermore, at point $s = 0$ one obtains

$$\begin{aligned}
\overline{B^2} &= \overline{S^2}(1 + \lambda \overline{B})^2 + \lambda \overline{S}\overline{B^2} \\
\overline{B^2}(1 - \lambda \overline{S}) &= \overline{S^2}(1 + \lambda \overline{B})^2 = \overline{S^2} \left(\frac{1 - \lambda \overline{S} + \lambda \overline{S}}{1 - \lambda \overline{S}}\right)^2 \\
\overline{B^2}(1 - \lambda \overline{S}) &= \frac{\overline{S^2}}{(1 - \lambda \overline{S})^2} \\
\overline{B^2} &= \frac{\overline{S^2}}{(1 - \lambda \overline{S})^3} = \frac{\overline{S^2}}{(1 - \rho)^3}.
\end{aligned}$$

EX 8: Time Reversibility, Queueing Networks

1. **Show that Kolmogorov criterion and detailed balance equations are equivalent in the three node network depicted on the right.**



Solution:

One needs to prove the equivalance of the following conditions:

$$\begin{array}{lllll} (K) & \text{Kolmogorov's criterion:} & q_{01} \cdot q_{12} \cdot q_{21} & = q_{02} \cdot q_{21} \cdot q_{10}, & \\ (D) & \text{Detailed balance:} & \pi_i \cdot q_{ij} & = \pi_j \cdot q_{ji}, & \forall\, i,j \end{array}$$

1° $(D) \Rightarrow (K)$
From (D) it follows that,

$$\begin{cases} \pi_0 q_{01} &= \pi_1 q_{10}, \\ \pi_1 q_{12} &= \pi_2 q_{21}, \\ \pi_2 q_{20} &= \pi_0 q_{02}. \end{cases}$$

Thus,

$$\pi_0 q_{01} \pi_1 q_{12} \pi_2 q_{20} = \pi_1 q_{10} \pi_2 q_{21} \pi_0 q_{02} \qquad \Rightarrow \qquad q_{01} q_{12} q_{20} = q_{10} q_{21} q_{02} \quad \Rightarrow (K).$$

2° $(K) \Rightarrow (D)$
Global balance equations,

$$\pi_0(q_{01} + q_{02}) = \pi_1 q_{10} + \pi_2 q_{20} \tag{8.1}$$
$$\pi_1(q_{10} + q_{12}) = \pi_0 q_{01} + \pi_2 q_{21} \tag{8.2}$$
$$\pi_2(q_{20} + q_{21}) = \pi_0 q_{02} + \pi_1 q_{12} \tag{8.3}$$

are always valid. We do a counter assumption that the detailed balance equations does not hold between certain states (i, j). Without limiting the generality, we can choose that,

$$\pi_0 q_{01} < \pi_1 q_{10},$$

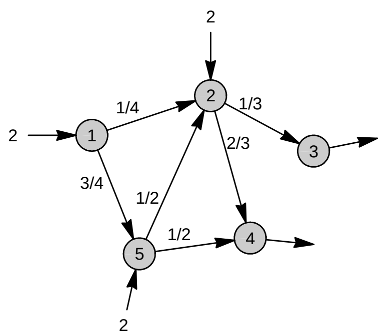
from which, together with (8.1) and (8.2), it follows that the detailed balance equation does not hold between the other states either:

$$\pi_2 q_{20} < \pi_0 q_{02},$$
$$\pi_1 q_{12} < \pi_2 q_{21}.$$

Thus,

$$\begin{array}{lrll} & \pi_0 q_{01} \cdot \pi_1 q_{12} \cdot \pi_2 q_{20} & < \pi_0 q_{02} \cdot \pi_2 q_{21} \cdot \pi_1 q_{10} & \\ \stackrel{(K)}{\Longrightarrow} & \pi_0 \cdot \pi_1 \cdot \pi_2 & < \pi_0 \cdot \pi_2 \cdot \pi_1 & \Longrightarrow \quad \text{not valid.} \end{array}$$

2. **Consiner the Jackson queueing network depicted below. Packets from outside arrive to the nodes 1, 2 and 5 as a Poisson stream with rate $\lambda = 2$ packets/s. In every node each link has own buffer. The incoming packet stream to each node is randomly directed with the depicted probabilities. The link from node 4 has capacity of $\mu = 8$ packets/s, while the capacity of the other links are $\mu = 3$ packets/s. a) What are the mean delays of packets taking the routes 1-2-3 and 1-5-4? b) How many packets there is on average in the network? c) What is the mean sojourn time of packets entering the network?**



Solution:

Waiting time (average delay) is $\overline{T} = \dfrac{\overline{N}}{\lambda} = \dfrac{\rho}{1-\rho}/\lambda = \dfrac{1/\mu}{1-\rho} = \dfrac{1}{\mu-\lambda}$.

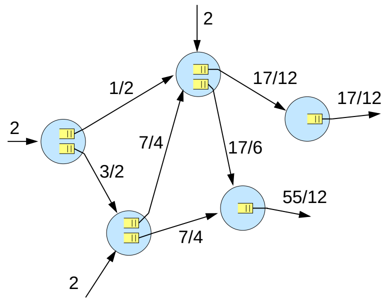


Figure 10: Traffic intensities in the network.

a) Mean delay in route 1-2-3 is,

$$E(T_{123}) = \frac{1}{3-1/2} + \frac{1}{3-17/12} + \frac{1}{3-17/12} \approx 1.66,$$

and mean delay in route 1-5-4 is,

$$E(T_{154}) = \frac{1}{3-3/2} + \frac{1}{3-7/4} + \frac{1}{8-55/12} \approx 1.76.$$

b) In the network there are on average $E(m) = \sum_{i=1}^{M} E(M_i)$ packets, where $E(m_i)$ is the number of packets in link i (either waiting on in service).
Now $E(m_i) = \lambda T_i = \frac{\lambda_i}{\mu_i - \lambda_i}$, and thus adding these together one obtains,

$$\begin{aligned}
E(m) &= \sum_{i=1}^{M} \lambda_i \mu_i - \lambda_i \\
&= \frac{1/2}{3-1/2} + \frac{3/2}{3-3/2} + \frac{7/4}{3-7/4} + \frac{17/12}{3-17/12} + \frac{17/6}{3-17/6} + \frac{7/4}{3-7/4}\frac{55/12}{8-55/12} + \frac{1}{8-55/12} \\
&= \frac{1}{6-1} + \frac{3}{6-3} + \frac{7}{12-7} + \frac{17}{36-17} + \frac{17}{18-17} + \frac{7}{12-7} + \frac{17}{36-17} + \frac{55}{96-55} \\
&= \frac{1}{5} + 1 + \frac{7}{5} + \frac{17}{19} + 17 + \frac{7}{5} + \frac{17}{19} + \frac{55}{41} = 21 + \frac{34}{19} + \frac{55}{41} \approx 24.13.
\end{aligned}$$

c) By applying the Little's result for the whole system one obtains

$$E(T) = \frac{E(m)}{\gamma}, \qquad \text{where } \gamma = 6\text{, and thus,} \qquad E(T) = \frac{24.13}{6} \approx \underline{4.02}.$$

Consider a cyclic closed network consisting of two queues. The service times in the queues are exponentially distributed with parameters μ_1 and μ_2. There are three customers circulating in the network. a) Draw the state transition diagram of the network (four states). b) Determine the equilibrium probabilities and calculate the mean queue lengths. c) Calculate the customer stream in the network (e.g. the customer stream departing from queue 1). d) Rederive the results of c) and d) by means of the mean value analysis (MVA).

Solution:

3. a) State diagram of the system is depicted in Fig. 11.

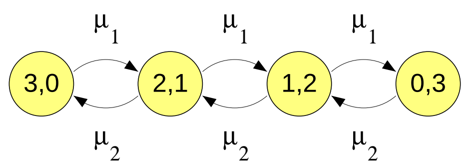


Figure 11: State diagram of the system.

b) In steady state it holds that

$$\begin{aligned} \mu_1\pi_0 &= \mu_2\pi_1 \\ \mu_1\pi_1 &= \mu_2\pi_2 \\ \mu_1\pi_2 &= \mu_2\pi_3 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \pi_1 &= \rho\pi_0 \\ \pi_2 &= \rho^2\pi_0 \\ \pi_3 &= \rho^3\pi_0 \end{aligned}$$

Normalization:

$$(1+\rho+\rho^2+\rho^3)\pi_0 = 1 \qquad \Rightarrow \qquad \pi_0 = \frac{1}{1+\rho+\rho^2+\rho^3}.$$

The average queue lengths are

$$\begin{aligned} N_1 &= (3+2\rho+\rho^2)\pi_0 = \frac{3+2\rho+\rho^2}{1+\rho+\rho^2+\rho^3} \\ N_2 &= (3\rho^3+2\rho^2+\rho)\pi_0 = \frac{3\rho^3+2\rho^2+\rho}{1+\rho+\rho^2+\rho^3}. \qquad (N_1+N_2=3) \end{aligned}$$

c) The traffic flow is

$$\lambda = (\pi_0+\pi_1+\pi_2)\mu_1 = \frac{1+\rho+\rho^2}{1+\rho+\rho^2+\rho^3}\mu_1 = \frac{1-\rho^3}{1-\rho}\frac{1-\rho}{1-\rho^4}\mu_1 = \frac{1-\rho^3}{1-\rho^4}\mu_1.$$

d) Mean value analysis (MVA):

$$\begin{cases} T_i[k] &= (1+N_i[k-1])\,/\mu_1 \\ N_i[k] &= k\frac{\lambda_i T_i[k]}{\sum_j \lambda_j T_j[k]} \\ \lambda_i[k] &= N_i[k]/T_i[k] \end{cases}$$

Here,

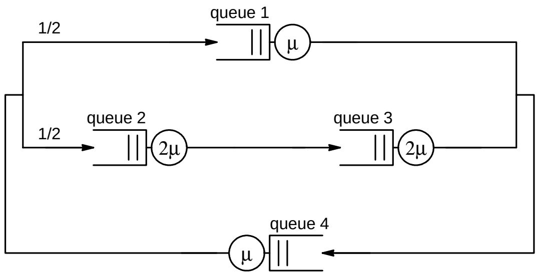
$$\begin{cases} N[0] &= [0,0] \\ T[1] &= 1/\mu_1\,[1,\rho] \\ N[1] &= \left[\frac{1}{1+\rho},\frac{\rho}{1+\rho}\right] \end{cases} \qquad \begin{cases} T[2] &= 1/\mu_1\left[\frac{2+\rho}{1+\rho},\frac{1+2\rho}{1+\rho}\rho\right] = \frac{1}{\mu_1(1+\rho)}\left[2+\rho,\rho+2\rho^2\right] \\ N[2] &= 2\left[\frac{2+\rho}{2+2\rho+2\rho^2},\frac{\rho+2\rho^2}{2+2\rho+2\rho^2}\right] = \left[\frac{2+\rho}{1+\rho+\rho^2},\frac{\rho+2\rho^2}{1+\rho+\rho^2}\right] \end{cases}$$

$$\begin{cases} T[3] &= 1/\mu_1\left[\frac{1+\rho+\rho^2+2+\rho}{1+\rho+\rho^2}, 1+\rho+\rho^2+\rho+2\rho^2 1+\rho+\rho^2\rho\right] \\ &= \frac{1}{\mu_1(1+\rho+\rho^2)}\left[3+2\rho+\rho^2,\rho+2\rho^2+3\rho^3\right] \\ N[3] &= 3\left[\frac{3+2\rho+\rho^2}{3+3\rho+3\rho^2+3\rho^3},\frac{\rho+2\rho^2+3\rho^3}{3+3\rho+3\rho^2+3\rho^3}\right] \\ &= \left[3+2\rho+\rho^2,\rho+2\rho^2+3\rho^3\right]\pi_0 \end{cases}$$

Similarly, the traffic flow becomes

$$\lambda = \lambda_1[3] = \frac{3+2\rho+\rho^2}{1+\rho+\rho^2+\rho^3}\frac{\mu_1(1+\rho+\rho^2)}{3+2\rho+\rho^2} = \frac{\mu_1(1+\rho+\rho^2)}{1+\rho+\rho^2+\rho^3} = \underline{\underline{\frac{1-\rho^3}{1-\rho^4}\mu_1}}.$$

4. **In the closed queueing network depicted in the figure there are four customers. What is the average cycle time of an customer and the average customer stream through queue 4?**



Solution:

In the lecture notes the following recursive method is presented. Initially we start with zero customers and in each step one customer is added. Let $\bar{N}_i[0] = 0$ and the recursive step is

$$\begin{aligned}
\bar{T}_i[k] &= (1 + \bar{N}_i[k-1])\frac{1}{\mu_i} \\
\bar{N}_i[k] &= k\frac{\hat{\lambda}_i \bar{T}_i[k]}{\sum_j \hat{\lambda}_j \bar{T}_j[k]} \\
\lambda_i[k] &= \bar{N}_i[k]/\bar{T}_i[k]
\end{aligned}$$

Here,

$$\begin{aligned}
\mu_1 &= \mu_4 = \mu & \qquad \lambda_1 &= \lambda_2 = \lambda_3 = \lambda/2 \\
\mu_2 &= \mu_3 = 2\mu & \qquad \lambda_4 &= \lambda
\end{aligned}$$

Matlab can be used to solve the problem numerically:

```
function r089( n )

mu = [ 1 2 2 1 ];
l  = [ 1 1 1 2 ];
N  = [ 0 0 0 0 ];

for k=1:4
    T   = ( [ 1 1 1 1 ] + N ) ./ mu;
    apu = sum ( l .* T );
    N   = k * ( l.* T ) / apu;
    [ T ; N ]
end;

lambda = N./T
k./ lambda
```

```
1.0000    0.5000    0.5000    1.0000
0.2500    0.1250    0.1250    0.5000

1.2500    0.5625    0.5625    1.5000
0.4651    0.2093    0.2093    1.1163

1.4651    0.6047    0.6047    2.1163
0.6364    0.2626    0.2626    1.8384

1.6364    0.6313    0.6313    2.8384
0.7633    0.2945    0.2945    2.6478

0.4664    0.4664    0.4664    0.9329
8.5758    8.5758    8.5758    4.2879
```

It holds that

$$\lambda_i[k] = \frac{\bar{N}_i[k]}{\bar{T}_i[k]}$$

so the average traffic streams of queues are

$$\lambda[4] = \mu(0.47, 0.47, 0.47, 0.93)$$

from which it can be deduced that the traffic flow of queue 4 is 0.93μ.

The average cycle time of customer through queue 4 is obtained by making *a cut* immediately after the queue 4. The traffic flow in and out of the cut are equal. The average delay from the moment when a packet leaves from the cut and returns to it is $N/\lambda_4 = 4/\lambda_4 \approx \frac{4.3}{\mu}$. The average cycle times of the other queues are two times longer.

5. **Consider Engset's system with n customers and s servers. Let X_i denote whether source i is on ($X_i = 1$) or off ($X_i = 0$). Without the limitation posed by the number of servers each source would change independently between the states 0 and 1 with transition rates γ and μ, so**

that the probability of state 1 is $p = \gamma/(\gamma+\mu)$. Show that each process X_i, $i = 1, \ldots, n$ is reversible and thus the joint process $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is also reversible (no need to prove). Furthermore, by truncation of a reversible process show that systems steady state probability distribution with a limited number of servers s is,

$$\mathrm{P}\{\mathbf{X} = \mathbf{j}\} = \begin{cases} C \prod_{i=1}^{n} p^{j_i}(1-p)^{1-j_i}, & \sum_{i=1}^{n} j_i \le s, \\ 0, & \text{otherwise}, \end{cases}$$

where j is a n-vector consisting of zeroes and ones, and C is a normalization constant.

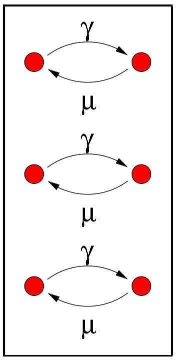
In the similar way one can write down steady state distribution for a smaller system $\mathbf{X}' = (X_2', \ldots, X_n')$, with $n-1$ sources (i.e. one source is missing) and the same number of servers. Show that steady state distribution of other components of $\mathbf{X}$ on condition that source 1 is off, is equal to the steady state distribution of $\mathbf{X}'$,

$$\mathrm{P}\{(X_2, \ldots, X_n) = \mathbf{j}' | X_1 = 0\} = \mathrm{P}\{\mathbf{X}' = \mathbf{j}'\},$$

where $\mathbf{j}'$ is again $(n-1)$-vector consisting of zeroes and ones. Based on that, deduce that the blocking probability experienced by source 1 is equal to the time blocking in the system with one source less.

Solution:

Let each X_i be independent of others, i.e. it is assumed that the number of servers not less than the number of sources. Then the detailed balance equation holds for each X_i,



$$\pi_0^{(i)} q_{0,1} = \pi_1^{(i)} q_{1,0}$$

where,

$$\begin{cases} q_{1,0} &= \mu, \\ q_{0,1} &= \gamma, \end{cases} \qquad \begin{cases} \pi_0^{(i)} &= \frac{\mu}{\gamma+\mu}, \\ \pi_1^{(i)} &= \frac{\gamma}{\gamma+\mu}. \end{cases}$$

Thus, each X_i is time reversible.

As each process is time reversible the joint process is also time reversible (assuming sufficient number of servers).

In the lecture notes it is shown that when a time reversible process is truncated the state probabilities of the resulting process are the same as in the original process multiplied with some normalization constant. In this case the state probabilities of the original process are clearly,

$$\mathrm{P}\{\mathbf{X} = \mathbf{j}\} = \prod_{i=1}^{n} p^{j_i}(1-p)^{1-j_i},$$

from what the claim follows.

Next we try to determine the state distribution of other components on condition that source 1 is in off state:

$$\begin{aligned} \mathrm{P}\{(X_2, \ldots, X_n) = \mathbf{j}' | X_1 = 0\} &= \frac{\mathrm{P}\{\mathbf{X} = (0, j_2, \ldots, j_n)\}}{\mathrm{P}\{X_1 = 0\}} \\ &= \frac{C(1-p)\prod_{i=2}^{n} p^{j_i}(1-p)^{1-j_i}}{C(1-p)\sum_{\mathbf{j}^* \in \mathbf{I}} \prod_{i=2}^{n} p^{j_i^*}(1-p)^{1-j_i^*}} \\ &= C' \prod_{i=1}^{n-1} p^{j_i'}(1-p)^{1-j_i'} = \mathrm{P}\{\mathbf{X}' = \mathbf{j}'\}. \end{aligned}$$

Consider next the call blocking probability experienced by source 1. Let,

$$\begin{cases} A &= A(t) &\leftrightarrow (X_2, \ldots, X_n) = \mathbf{j}', \\ B &= B(t) &\leftrightarrow X_1(t) = 0, \\ C &= C(t, t+dt) &\leftrightarrow \text{lähde 1 yrittää mennä päälle aikavälillä } (t, t+dt)\ . \end{cases}$$

The state probability distribution of other components seen by source 1 when it tries to move from OFF state to ON state is,

$$\begin{aligned} \mathrm{P}\{A|C\} &= \mathrm{P}\{A|C, B\} \\ &= \frac{\mathrm{P}\{A, C, B\}}{\mathrm{P}\{C, B\}} \\ &= \frac{\mathrm{P}\{C|A, B\} \cdot \mathrm{P}\{A, B\}}{\mathrm{P}\{C|B\} \cdot \mathrm{P}\{B\}} \\ &= \frac{\mathrm{P}\{A, B\}}{\mathrm{P}\{B\}} = \mathrm{P}\{A|B\} = \mathrm{P}\{\mathbf{X}' = \mathbf{j}'\}. \end{aligned}$$

Thus the state probability distribution of other components seen by source 1 when it tries to move to the on state is equal to the steady state distribution of smaller system. Hence, the call blocking probability of source 1 is equal to the time blocking in the smaller system.

A Some Formulas and Tables

Polynomials and series

binomial theorem	$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$
factorizations	$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \ldots - ab^{n-2} + b^{n-1})$, $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})$
series	$\sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x \qquad \sum_{i=0}^{\infty} q^i = \frac{1}{1-q}, \quad \lvert q\rvert < 1$ $\sum_{i=0}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Power series

Power series $S(z) = \sum_{i=0}^{\infty} a_i z^i$ converge for all $|z| < R$, where R the radius of convergence: $R = \lim_{i\to\infty} \left|\frac{a_i}{a_{i+1}}\right|$ or $R = \lim_{i\to\infty} \frac{1}{\sqrt[i]{|a_i|}}$ (if exists). The serie can be integrated and differentiated term by term inside the radius of convergence R, $S'(z) = \sum_{i=1}^{n} i a_i z^{i-1} < \infty$.

Discrete Distributions

name	$\mathrm{P}\{X = i\}$		generating fn.	$\mathrm{E}[X]$	σ^2
Geometric	$(1-p)^{i-1}p$	$i = 1, 2, 3, \ldots$	$\frac{pz}{1-(1-p)z}$	$1/p$	$(1-p)/p^2$
Poisson	$\frac{\lambda^i}{i!}e^{-\lambda}$	$i = 0, 1, 2, \ldots$	$e^{\lambda(z-1)}$	λ	λ
Binomial	$\binom{n}{i}p^i(1-p)^{n-i}$	$i = 0, 1, \ldots, n$	$(1-p+pz)^n$	np	$np(1-p)$
Bernoulli	$\begin{cases} p & \text{when } i = 1 \\ 1-p & \text{when } i = 0 \end{cases}$	$i = 0, 1$	$1-p+pz$	p	$p(1-p)$
Uniform	$1/n$	$i = 1, \ldots, n$	$\frac{z-z^{n+1}}{n-nz}$	$\frac{n+1}{2}$	$\frac{(k+1)(2k+1)}{6}$

Continuous Distributions

name	density function		generating fn.	$\mathrm{E}[X]$	σ^2
Exponential	$\lambda e^{-\lambda x}$	$\lambda, x > 0$	$(1-t/\lambda)^{-1}$	$1/\lambda$	$1/\lambda^2$
Normal	$\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\sigma > 0, -\infty < x < \infty,$ $-\infty < \mu < \infty$	$e^{\mu t + \sigma^2 t^2/2}$	μ	σ^2

Probability Calculus

name	formula	explanation
Bayes formula	$\mathrm{P}\{A_j \mid B\} = \frac{\mathrm{P}\{B \mid A_j\}\mathrm{P}\{A_j\}}{\sum_i \mathrm{P}\{B \mid A_i\}\mathrm{P}\{A_i\}}$	$\bigcup_i A_i = S$ and $A_i \bigcap A_j = \emptyset \ \forall i \neq j$
De Morgan	$A \cup B = \left(A^C \cap B^C\right)^C$, $A \cap B = \left(A^C \cup B^C\right)^C$	
Markov's inequality	$\mathrm{P}\{X \geq a\} \leq \mu/a$	X non-negative, $a > 0$
Chebyshev's inequality	$\mathrm{P}\{\lvert X-\mu\rvert \geq k\} \leq \sigma^2/k^2$	

Generating function, z-transformation

Let X be a non-negative integer valued probability variable: $\mathrm{P}\{X = i\} = p_i,$ when $i = 0, 1, \ldots$.

The generating function of X, denoted by $\mathcal{G}_X$, is, $\mathcal{G}_X(z) = \sum_{i=0}^{\infty} p_i\, z^i = \mathrm{E}\left[z^X\right].$

Y	$\mathcal{G}_Y$	note
$X_1 + X_2 + \ldots + X_N$	$\mathcal{G}_N(\mathcal{G}_X(z))$	$X_i \sim X$, random sum
$A + B$	$\mathcal{G}_A(z) \cdot \mathcal{G}_B(z)$	A and B are independent

Twisted distributions

Denote with X_β a twisted random variable obtained from random variable X, where β is the twisting parameter. Then the pdf of X_β is

$$f_\beta(x) = \frac{e^{\beta x} f(x)}{M(\beta)},$$

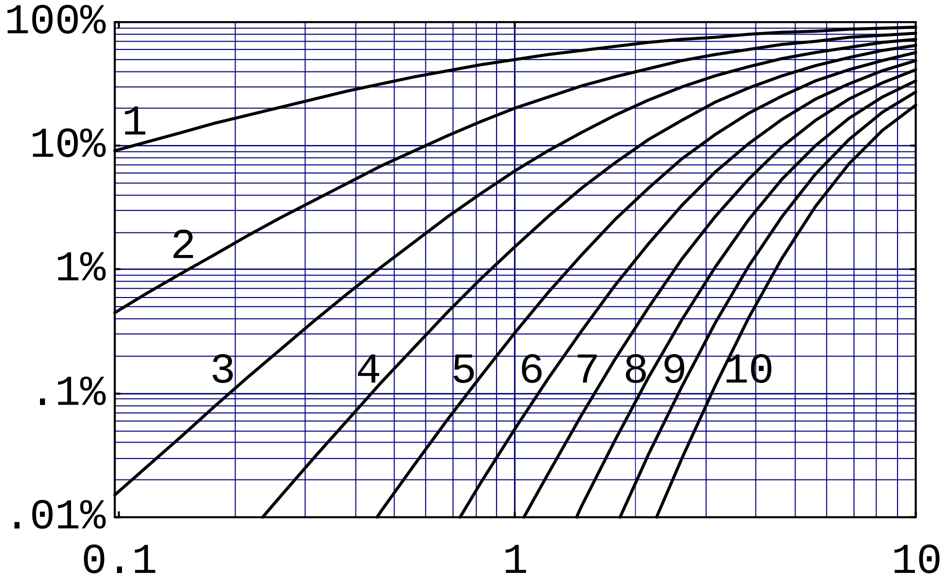
where $M(\beta)$ is the moment generating function of X, i.e. $M(\beta) = \mathrm{E}\left[e^{\beta X}\right]$.

Queueing Theory

name	formula	explanation
Little's formula	$\overline{N} = \lambda \cdot \overline{T}$	N=number of customers in system, λ=arrival intensity and T=service time
M/M/1-queue	$\pi_i = (1-\rho)\rho^i$ $\overline{N} = \dfrac{\rho}{1-\rho}$	$\rho = \lambda/\mu$, $\rho < 1$
M/G/1-queue	$\overline{W} = \dfrac{\lambda\overline{S^2}}{2(1-\lambda\overline{S})}, \quad \overline{T} = \overline{S} + \overline{W}$	S is the service time
Erlangs B-formula	$E(n,a) = \dfrac{a^n/n!}{1 + a/1! + \ldots + a^n/n!}$	n=servers, a=load

Erlang's blocking formula

$$\boxed{E(n,a) = \frac{a^n/n!}{1 + a/1! + \ldots + a^n/n!}}$$



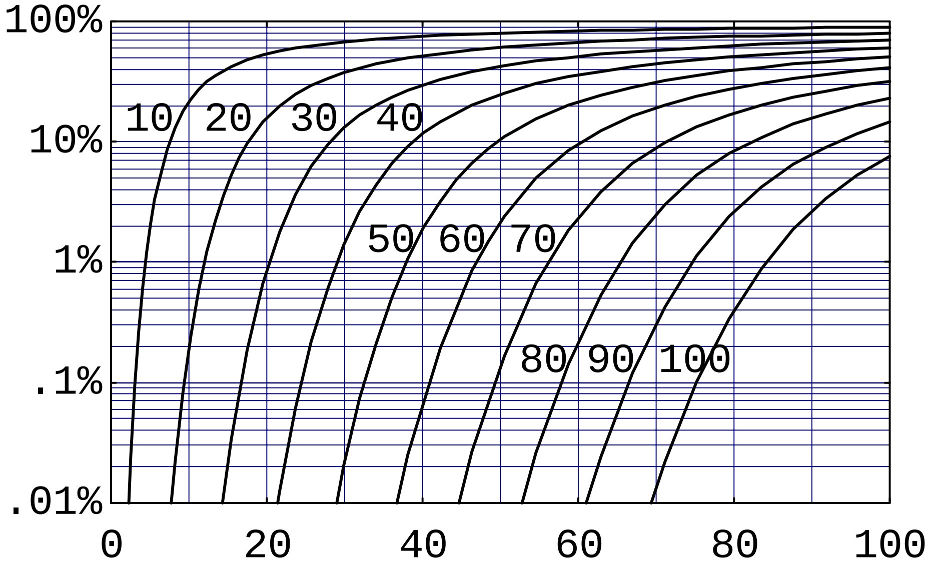


Figure 12: Erlang blocking formula $E(n, A)$ as a function of the offered traffic intensity A. The number of servers n is the parameter of the family of curves: upper figure $n = 1, \ldots 10$, lower figure $n = 10, \ldots 100$.

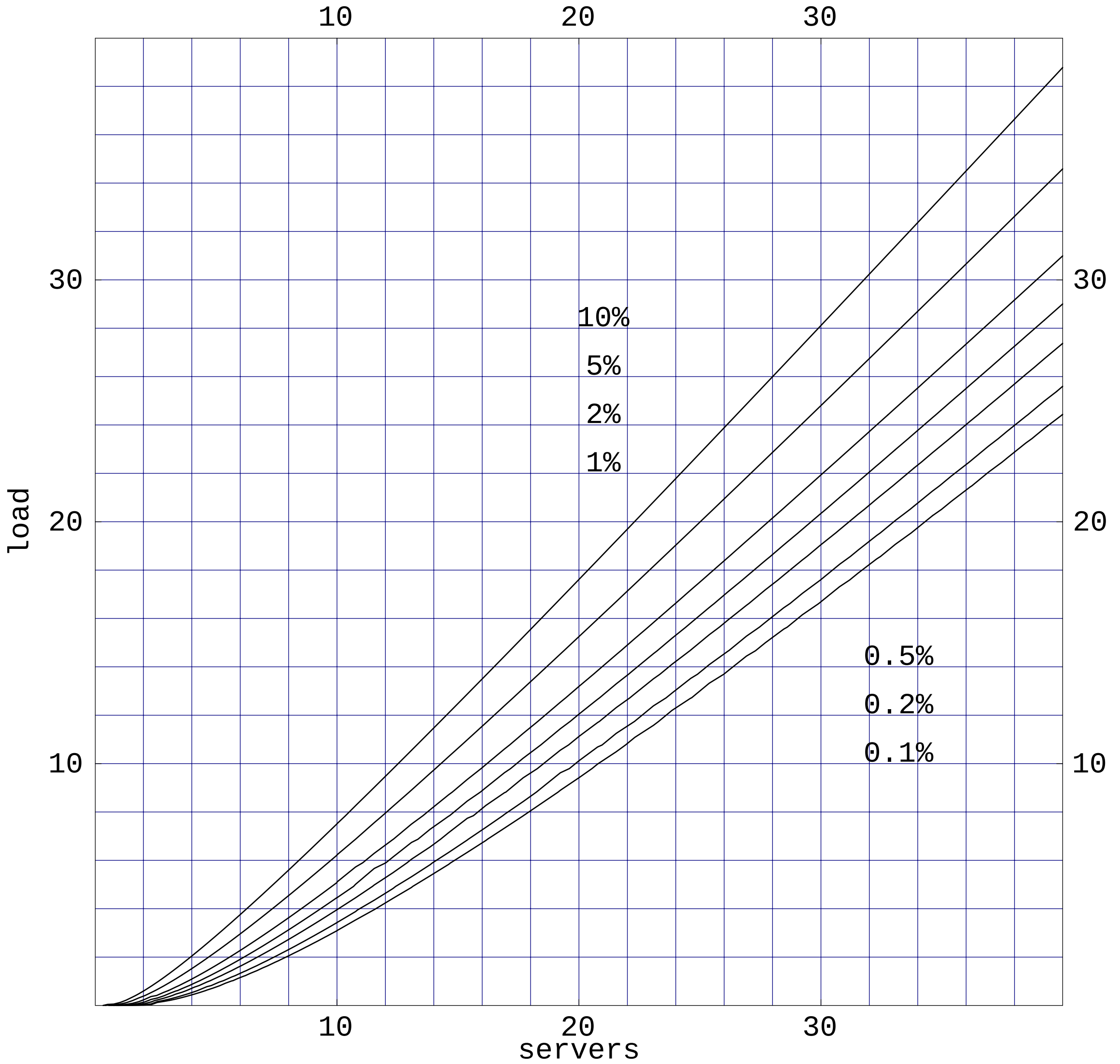


Figure 13: The profiles of Erlang blocking formula $E(n, A)$ for 0.1%, 0.2%, 0.5%, 1%, 2%, 5% and 10% blocking probability.

Example source code

Mathematica

```
BeginPackage["tlt`"];

(*
 * Commonly used functions in S-38141 Teletraffic Theory
 *)

(* Erlang's blocking formula *)
erl::usage  = "erl[ n, a ], Erlang's B-formula, n servers and load a.";
erld::usage = "erld[ x, a ], generalized Erlang's B-formula for real values.";
erli::usage = "erli[ a, B ], inverse Erlang's formula, how many servers required.";

(* Markov Chains *)
DTMC::usage  = "DTMC[P], steady state distribution of discrete time Markov chain.";
CTMC::usage  = "CTMC[Q], steady state distribution of continuous time Markov chain.";

(* other *)
Qndd1::usage = "Qndd1[x, N, D], virtual waiting time distribution for the N*D/D/1 queue.";

Begin["`Private`"];

erl[n_,a_]    := Module[{e=1}, Do[e=1+i*e/a,{i,n}]; 1/e]
erld[x_, a_] := N[a^x Exp[-a]/Gamma[x+1,a]] /; (x >= 0) && (a > 0)
erli[a_,B_]   := Module[{e=1,r=1/B,n=0}, While[(e=1+n*e/a)<r,n++];n]

CTMC[Q_]       := Module[{n=Length[Q],EE,ee},
                    EE=Table[1,{n},{n}];
                    ee=Table[1,{n}];
                    Chop[ee.Inverse[Q+EE]]]

DTMC[P_]       := Module[{n=Length[P],EE,ee},
                    EE=Table[1,{n},{n}]-IdentityMatrix[n];
                    ee=Table[1,{n}];
                    Chop[ee.Inverse[P+EE]]]

Qndd1[x_, N_, D_] := Module[{sum = 0, bin = 1, n},
                    For[n = N, n > x, n--,
                      sum += bin ((n-x)/D)^n (1-(n-x)/D)^(N-n) (D-N+x)/(D-n+x);
                      bin *= n/(N-n+1);
                    ];
                    sum ]

End[];
EndPackage[];
```

Matlab

```
function b = erl( n, a )
% Erlang's blocking formula, erl(n,a)
e = 1;
for i = 1:n,
  e = 1 + i*e/a;
end
b = 1/e;
```

```
function b = erld( x, a )
% Erlang's (generalized) blocking formula, erld(x,a)
b = a^x * exp(-a) / ( gamma(x+1)*(1 - gammainc(a,x+1) ) );
```

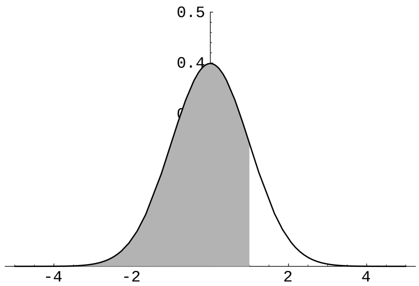
```
function n = erli( a, B )
% Inverse Erlang's blocking formula, erli(a,B)
n = 0;
r = 1/B;
e = 1 + n/a;
while e < r,
  n = n + 1;
  e = 1 + n*e/a;
```

```
end
```

```
function p = DTMC( P )
% DTMC( P ), returns steady state distribution of a Discrete Time Markov Chain
n = length( P );
p = ones( 1, n ) * inv( P + ones(n) - eye(n) );
```

```
function p = CTMC( Q )
% CTMC( Q ), returns steady state distribution of a Continuous Time Markov Chain
n = length( Q );
p = ones( 1, n ) * inv( Q + ones( n ) );
```

Normal Distribution Table



$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{t^2/2}\, dt$$

$$\Phi(x) + \Phi(-x) = 1$$

x	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7793	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8943	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Approximation for the large values of x,

$$1 - \Phi(x) \approx \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}.$$

Index